

Exercise 2.5.3

Solve Laplace's equation *outside* a circular disk ($r \geq a$) subject to the boundary condition [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r, \theta) = \phi(\theta)G(r)$, then $\frac{r}{G} \frac{d}{dr} \left(r \frac{dG}{dr} \right) = -\frac{1}{\phi} \frac{d^2 \phi}{d\theta^2}$]:

(a) $u(a, \theta) = \ln 2 + 4 \cos 3\theta$

(b) $u(a, \theta) = f(\theta)$

You may assume that $u(r, \theta)$ remains finite as $r \rightarrow \infty$.

Solution

Because the boundary condition of the Laplace equation is prescribed on a circle, the method of separation of variables can be applied. Assume a product solution of the form $u(r, \theta) = R(r)\Theta(\theta)$ and substitute it into the PDE.

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \rightarrow \quad \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} [R(r)\Theta(\theta)] \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} [R(r)\Theta(\theta)] = 0$$

Proceed to separate variables.

$$\frac{\Theta}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{R}{r^2} \frac{d^2 \Theta}{d\theta^2} = 0$$

Multiply both sides by $r^2/[R(r)\Theta(\theta)]$.

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = 0$$

Bring the second term to the right side. (The final answer will be the same regardless which side the minus sign is on.)

$$\underbrace{\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right)}_{\text{function of } r} = - \underbrace{\frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2}}_{\text{function of } \theta}$$

The only way a function of r can be equal to a function of θ is if both are equal to a constant λ .

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = - \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = \lambda$$

As a result of applying the method of separation of variables, the Laplace equation has been reduced to two ODEs—one in r and one in θ .

$$\left. \begin{aligned} \frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) &= \lambda \\ - \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} &= \lambda \end{aligned} \right\}$$

Periodic boundary conditions are assumed for Θ , since the solution and its slope in the θ -direction are expected to be the same at $\theta = 0$ and $\theta = 2\pi$.

$$\begin{aligned}\Theta(0) &= \Theta(2\pi) \\ \frac{d\Theta}{d\theta}(0) &= \frac{d\Theta}{d\theta}(2\pi)\end{aligned}$$

Values of λ for which nontrivial solutions of the preceding equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for Θ becomes

$$\frac{d^2\Theta}{d\theta^2} = -\alpha^2\Theta.$$

The general solution is written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos \alpha\theta + C_2 \sin \alpha\theta$$

Take the derivative of it.

$$\Theta'(\theta) = \alpha(-C_1 \sin \alpha\theta + C_2 \cos \alpha\theta)$$

Apply the boundary conditions to obtain a system of equations involving C_1 and C_2 .

$$\begin{aligned}\Theta(0) &= C_1 = C_1 \cos 2\pi\alpha + C_2 \sin 2\pi\alpha = \Theta(2\pi) \\ \Theta'(0) &= \alpha(C_2) = \alpha(-C_1 \sin 2\pi\alpha + C_2 \cos 2\pi\alpha) = \Theta'(2\pi)\end{aligned}$$

$$\begin{cases} C_1 = C_1 \cos 2\pi\alpha + C_2 \sin 2\pi\alpha \\ C_2 = -C_1 \sin 2\pi\alpha + C_2 \cos 2\pi\alpha \end{cases}$$

$$\begin{cases} C_1(1 - \cos 2\pi\alpha) = C_2 \sin 2\pi\alpha \\ C_2(1 - \cos 2\pi\alpha) = -C_1 \sin 2\pi\alpha \end{cases}$$

These equations are satisfied if $\alpha = n$, where $n = 1, 2, \dots$. The positive eigenvalues are thus $\lambda = n^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_1 \cos \alpha\theta + C_2 \sin \alpha\theta \quad \rightarrow \quad \Theta_n(\theta) = C_1 \cos n\theta + C_2 \sin n\theta.$$

With this formula for λ , the ODE for R becomes

$$\frac{r}{R} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = n^2$$

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - n^2 R = 0.$$

This ODE is equidimensional, so the general solution is of the form $R(r) = r^k$. Find its derivatives

$$R(r) = r^k \quad \rightarrow \quad \frac{dR}{dr} = kr^{k-1} \quad \rightarrow \quad \frac{d^2 R}{dr^2} = k(k-1)r^{k-2}$$

and substitute them into the equation.

$$r^2 k(k-1)r^{k-2} + rkr^{k-1} - n^2 r^k = 0$$

$$k(k-1)r^k + kr^k - n^2r^k = 0$$

Divide both sides by r^k .

$$k(k-1) + k - n^2 = 0$$

$$k^2 - n^2 = 0$$

$$k = \pm n$$

Consequently,

$$R(r) = C_3r^{-n} + C_4r^n.$$

Since u remains finite as $r \rightarrow \infty$, we require that $C_4 = 0$.

$$R(r) = \frac{C_3}{r^n}$$

Suppose secondly that λ is zero: $\lambda = 0$. The ODE for Θ becomes

$$\Theta'' = 0.$$

Integrate both sides with respect to θ .

$$\Theta' = C_5$$

Integrate both sides with respect to θ once more.

$$\Theta(\theta) = C_5\theta + C_6$$

Apply the boundary conditions to obtain a system of equations involving C_5 and C_6 .

$$\Theta(0) = C_6 = 2\pi C_5 + C_6 = \Theta(2\pi)$$

$$\Theta'(0) = C_5 = C_5 = \Theta'(2\pi)$$

The first equation implies that $C_5 = 0$ and C_6 is arbitrary, and the second equation gives no information.

$$\Theta(\theta) = C_6$$

Since $\Theta(\theta)$ is nonzero, zero is an eigenvalue; the eigenfunction associated with it is $\Theta_0(\theta) = 1$. Now solve the ODE for R with $\lambda = 0$.

$$\frac{d}{dr} \left(r \frac{dR}{dr} \right) = 0$$

Integrate both sides with respect to r .

$$r \frac{dR}{dr} = C_7$$

Divide both sides by r .

$$\frac{dR}{dr} = \frac{C_7}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = C_7 \ln r + C_8$$

For u to remain finite as $r \rightarrow \infty$, we require that $C_7 = 0$.

$$R(r) = C_8$$

Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for Θ becomes

$$\frac{d^2\Theta}{d\theta^2} = \beta^2\Theta.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_9 \cosh \beta\theta + C_{10} \sinh \beta\theta$$

Take the derivative of it.

$$\Theta'(\theta) = \beta(C_9 \sinh \beta\theta + C_{10} \cosh \beta\theta)$$

Apply the boundary conditions to obtain a system of equations involving C_9 and C_{10} .

$$\begin{aligned}\Theta(0) &= C_9 = C_9 \cosh 2\pi\beta + C_{10} \sinh 2\pi\beta = \Theta(2\pi) \\ \Theta'(0) &= \beta(C_{10}) = \beta(C_9 \sinh 2\pi\beta + C_{10} \cosh 2\pi\beta) = \Theta'(2\pi)\end{aligned}$$

$$\begin{cases} C_9 = C_9 \cosh 2\pi\beta + C_{10} \sinh 2\pi\beta \\ C_{10} = C_9 \sinh 2\pi\beta + C_{10} \cosh 2\pi\beta \end{cases}$$

$$\begin{cases} C_9(1 - \cosh 2\pi\beta) = C_{10} \sinh 2\pi\beta \\ C_{10}(1 - \cosh 2\pi\beta) = C_9 \sinh 2\pi\beta \end{cases}$$

No nonzero value of β satisfies these equations, so $C_9 = 0$ and $C_{10} = 0$. The trivial solution $\Theta(\theta) = 0$ is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE for u is a linear combination of $R(r)\Theta(\theta)$ over all the eigenvalues.

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} \frac{1}{r^n} (A_n \cos n\theta + B_n \sin n\theta)$$

Part (a)

Apply the boundary condition $u(a, \theta) = \ln 2 + 4 \cos 3\theta$ to determine the coefficients, A_0 , A_n , and B_n .

$$u(a, \theta) = A_0 + \sum_{n=1}^{\infty} \frac{1}{a^n} (A_n \cos n\theta + B_n \sin n\theta) = \ln 2 + 4 \cos 3\theta$$

By inspection, we see that $A_0 = \ln 2$, $B_n = 0$, and

$$\frac{A_n}{a^n} = \begin{cases} 0 & n \neq 3 \\ 4 & n = 3 \end{cases} \quad \rightarrow \quad A_n = \begin{cases} 0 & n \neq 3 \\ 4a^3 & n = 3 \end{cases}.$$

Therefore,

$$u(r, \theta) = \ln 2 + \frac{4a^3}{r^3} \cos 3\theta.$$

Part (b)

Apply the boundary condition $u(a, \theta) = f(\theta)$ to determine the coefficients, A_0 , A_n , and B_n .

$$u(a, \theta) = A_0 + \sum_{n=1}^{\infty} \frac{1}{a^n} (A_n \cos n\theta + B_n \sin n\theta) = f(\theta) \quad (1)$$

To find A_0 , integrate both sides of equation (1) with respect to θ from 0 to 2π .

$$\int_0^{2\pi} \left[A_0 + \sum_{n=1}^{\infty} \frac{1}{a^n} (A_n \cos n\theta + B_n \sin n\theta) \right] d\theta = \int_0^{2\pi} f(\theta) d\theta$$

Split up the integral on the left and bring the constants in front.

$$A_0 \int_0^{2\pi} d\theta + \sum_{n=1}^{\infty} \frac{1}{a^n} \left(\underbrace{A_n \int_0^{2\pi} \cos n\theta d\theta}_{=0} + \underbrace{B_n \int_0^{2\pi} \sin n\theta d\theta}_{=0} \right) = \int_0^{2\pi} f(\theta) d\theta$$

$$A_0(2\pi) = \int_0^{2\pi} f(\theta) d\theta$$

So then

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta.$$

To find A_n , multiply both sides of equation (1) by $\cos m\theta$, where m is an integer,

$$A_0 \cos m\theta + \sum_{n=1}^{\infty} \frac{1}{a^n} (A_n \cos n\theta \cos m\theta + B_n \sin n\theta \cos m\theta) = f(\theta) \cos m\theta$$

and then integrate both sides with respect to θ from 0 to 2π .

$$\int_0^{2\pi} \left[A_0 \cos m\theta + \sum_{n=1}^{\infty} \frac{1}{a^n} (A_n \cos n\theta \cos m\theta + B_n \sin n\theta \cos m\theta) \right] d\theta = \int_0^{2\pi} f(\theta) \cos m\theta d\theta$$

Split up the integral on the left and bring the constants in front.

$$A_0 \underbrace{\int_0^{2\pi} \cos m\theta d\theta}_{=0} + \sum_{n=1}^{\infty} \frac{1}{a^n} \left(\underbrace{A_n \int_0^{2\pi} \cos n\theta \cos m\theta d\theta}_{=0} + \underbrace{B_n \int_0^{2\pi} \sin n\theta \cos m\theta d\theta}_{=0} \right) = \int_0^{2\pi} f(\theta) \cos m\theta d\theta$$

Because the cosine functions are orthogonal, the second integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$\frac{A_n}{a^n} \int_0^{2\pi} \cos^2 n\theta d\theta = \int_0^{2\pi} f(\theta) \cos n\theta d\theta$$

$$\frac{A_n}{a^n} (\pi) = \int_0^{2\pi} f(\theta) \cos n\theta d\theta$$

So then

$$A_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta.$$

To find B_n , multiply both sides of equation (1) by $\sin m\theta$, where m is an integer,

$$A_0 \sin m\theta + \sum_{n=1}^{\infty} \frac{1}{a^n} (A_n \cos n\theta \sin m\theta + B_n \sin n\theta \sin m\theta) = f(\theta) \sin m\theta$$

and then integrate both sides with respect to θ from 0 to 2π .

$$\int_0^{2\pi} \left[A_0 \sin m\theta + \sum_{n=1}^{\infty} \frac{1}{a^n} (A_n \cos n\theta \sin m\theta + B_n \sin n\theta \sin m\theta) \right] d\theta = \int_0^{2\pi} f(\theta) \sin m\theta \, d\theta$$

Split up the integral on the left and bring the constants in front.

$$\begin{aligned} A_0 \underbrace{\int_0^{2\pi} \sin m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} \frac{1}{a^n} \left(A_n \underbrace{\int_0^{2\pi} \cos n\theta \sin m\theta \, d\theta}_{=0} + B_n \int_0^{2\pi} \sin n\theta \sin m\theta \, d\theta \right) \\ = \int_0^{2\pi} f(\theta) \sin m\theta \, d\theta \end{aligned}$$

Because the sine functions are orthogonal, the third integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$\frac{B_n}{a^n} \int_0^{2\pi} \sin^2 n\theta \, d\theta = \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

$$\frac{B_n}{a^n} (\pi) = \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

So then

$$B_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta.$$