Exercise 2.5.4

For Laplace’s equation inside a circular disk \((r \leq a)\), using (2.5.45) and (2.5.47), show that
\[
   u(r, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tilde{\theta}) \left[ -\frac{1}{2} + \sum_{n=0}^{\infty} \left( \frac{r}{a} \right)^n \cos n(\theta - \tilde{\theta}) \right] \, d\tilde{\theta}.
\]

Using \(\cos z = \text{Re} \left[ e^{iz} \right]\), sum the resulting geometric series to obtain Poisson’s integral formula.

**Solution**

Here we will solve the Laplace equation inside a disk.

\[
   \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0, \quad r \leq a, \quad 0 \leq \theta \leq 2\pi
\]

\(u(a, \theta) = f(\theta)\)

\(u(r, \theta)\) bounded as \(r \to 0\)

Because the boundary condition of the Laplace equation is prescribed on a circle, the method of separation of variables can be applied. Assume a product solution of the form \(u(r, \theta) = R(r)\Theta(\theta)\) and substitute it into the PDE.

\[
   \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \rightarrow \quad \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} [R(r)\Theta(\theta)] \right] + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} [R(r)\Theta(\theta)] = 0
\]

Proceed to separate variables.

\[
   \frac{\Theta}{r} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{R d^2 \Theta}{r^2 \Theta d^2 \Theta} = 0
\]

Multiply both sides by \(r^2/[R(r)\Theta(\theta)]\).

\[
   \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{1}{\Theta} d^2 \Theta = 0
\]

Bring the second term to the right side. (The final answer will be the same regardless which side the minus sign is on.)

\[
   \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) = -\frac{1}{\Theta} d^2 \Theta
\]

The only way a function of \(r\) can be equal to a function of \(\theta\) is if both are equal to a constant \(\lambda\).

\[
   \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) = -\frac{1}{\Theta} d^2 \Theta = \lambda
\]

As a result of applying the method of separation of variables, the Laplace equation has been reduced to two ODEs—one in \(r\) and one in \(\theta\).

\[
   \begin{cases}
      \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) = \lambda \\
      \frac{1}{\Theta} d^2 \Theta = \lambda
   \end{cases}
\]

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Periodic boundary conditions are assumed for $\Theta$, since the solution and its slope in the $\theta$-direction are expected to be the same at $\theta = 0$ and $\theta = 2\pi$.

$$\Theta(0) = \Theta(2\pi)$$
$$\frac{d\Theta}{d\theta}(0) = \frac{d\Theta}{d\theta}(2\pi)$$

Values of $\lambda$ for which nontrivial solutions of the preceding equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that $\lambda$ is positive: $\lambda = \alpha^2$. The ODE for $\Theta$ becomes

$$\frac{d^2\Theta}{d\theta^2} = -\alpha^2 \Theta.$$

The general solution is written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos \alpha \theta + C_2 \sin \alpha \theta$$

Take the derivative of it.

$$\Theta'(\theta) = \alpha (-C_1 \sin \alpha \theta + C_2 \cos \alpha \theta)$$

Apply the boundary conditions to obtain a system of equations involving $C_1$ and $C_2$.

$$\Theta(0) = C_1 = C_1 \cos 2\pi \alpha + C_2 \sin 2\pi \alpha = \Theta(2\pi)$$
$$\Theta'(0) = \alpha (C_2) = \alpha (-C_1 \sin 2\pi \alpha + C_2 \cos 2\pi \alpha) = \Theta'(2\pi)$$

$$\begin{cases}
C_1 = C_1 \cos 2\pi \alpha + C_2 \sin 2\pi \alpha \\
C_2 = -C_1 \sin 2\pi \alpha + C_2 \cos 2\pi \alpha
\end{cases}$$

$$\begin{cases}
C_1 (1 - \cos 2\pi \alpha) = C_2 \sin 2\pi \alpha \\
C_2 (1 - \cos 2\pi \alpha) = -C_1 \sin 2\pi \alpha
\end{cases}$$

These equations are satisfied if $\alpha = n$, where $n = 1, 2, \ldots$. The positive eigenvalues are thus $\lambda = n^2$, and the eigenfunctions associated with them are

$$\Theta(n \theta) = C_1 \cos n \theta + C_2 \sin n \theta.$$

With this formula for $\lambda$, the ODE for $R$ becomes

$$\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) = n^2$$

$$r^2 \frac{d^2R}{dr^2} + r \frac{dR}{dr} - n^2 R = 0.$$

This ODE is equidimensional, so the general solution is of the form $R(r) = r^k$. Find its derivatives

$$R(r) = r^k \quad \rightarrow \quad \frac{dR}{dr} = kr^{k-1} \quad \rightarrow \quad \frac{d^2R}{dr^2} = k(k-1)r^{k-2}$$

and substitute them into the equation.

$$r^2 k(k-1)r^{k-2} + kr^{k-1} - n^2 r^k = 0$$

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\[ k(k - 1)r^k + kr^k - n^2r^k = 0 \]

Divide both sides by \( r^k \).
\[ k(k - 1) + k - n^2 = 0 \]
\[ k^2 - n^2 = 0 \]
\[ k = \pm n \]

Consequently,
\[ R(r) = C_3 r^{-n} + C_4 r^n. \]

Since \( u \) remains finite as \( r \to 0 \), we require that \( C_3 = 0 \).
\[ R(r) = C_4 r^n \]

Suppose secondly that \( \lambda \) is zero: \( \lambda = 0 \). The ODE for \( \Theta \) becomes
\[ \Theta'' = 0. \]

Integrate both sides with respect to \( \theta \).
\[ \Theta' = C_5 \]

Integrate both sides with respect to \( \theta \) once more.
\[ \Theta(\theta) = C_5 \theta + C_6 \]

Apply the boundary conditions to obtain a system of equations involving \( C_5 \) and \( C_6 \).
\[ \Theta(0) = C_6 = 2\pi C_5 + C_6 = \Theta(2\pi) \]
\[ \Theta'(0) = C_5 = C_5 = \Theta'(2\pi) \]

The first equation implies that \( C_5 = 0 \) and \( C_6 \) is arbitrary, and the second equation gives no information.
\[ \Theta(\theta) = C_6 \]

Since \( \Theta(\theta) \) is nonzero, zero is an eigenvalue; the eigenfunction associated with it is \( \Theta_0(\theta) = 1 \).

Now solve the ODE for \( R \) with \( \lambda = 0 \).
\[ \frac{d}{dr} \left( r \frac{dR}{dr} \right) = 0 \]

Integrate both sides with respect to \( r \).
\[ r \frac{dR}{dr} = C_7 \]

Divide both sides by \( r \).
\[ \frac{dR}{dr} = \frac{C_7}{r} \]

Integrate both sides with respect to \( r \) once more.
\[ R(r) = C_7 \ln r + C_8 \]

For \( u \) to remain finite as \( r \to 0 \), we require that \( C_7 = 0 \).
\[ R(r) = C_8 \]
Suppose thirdly that \( \lambda \) is negative: \( \lambda = -\beta^2 \). The ODE for \( \Theta \) becomes

\[
\frac{d^2 \Theta}{d\theta^2} = \beta^2 \Theta.
\]

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

\[
\Theta(\theta) = C_9 \cosh \beta \theta + C_{10} \sinh \beta \theta
\]

Take the derivative of it.

\[
\Theta'(\theta) = \beta (C_9 \sinh \beta \theta + C_{10} \cosh \beta \theta)
\]

Apply the boundary conditions to obtain a system of equations involving \( C_9 \) and \( C_{10} \).

\[
\begin{aligned}
\Theta(0) = C_9 &= C_9 \cosh 2\pi \beta + C_{10} \sinh 2\pi \beta = \Theta(2\pi) \\
\Theta'(0) = \beta (C_{10}) &= \beta (C_9 \sinh 2\pi \beta + C_{10} \cosh 2\pi \beta) = \Theta'(2\pi)
\end{aligned}
\]

\[
\begin{cases}
C_9 = C_9 \cosh 2\pi \beta + C_{10} \sinh 2\pi \beta \\
C_{10} = C_9 \sinh 2\pi \beta + C_{10} \cosh 2\pi \beta
\end{cases}
\]

\[
\begin{cases}
C_9 (1 - \cosh 2\pi \beta) = C_{10} \sinh 2\pi \beta \\
C_{10} (1 - \cosh 2\pi \beta) = C_9 \sinh 2\pi \beta
\end{cases}
\]

No nonzero value of \( \beta \) satisfies these equations, so \( C_9 = 0 \) and \( C_{10} = 0 \). The trivial solution \( \Theta(\theta) = 0 \) is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE for \( u \) is a linear combination of \( R(r) \Theta(\theta) \) over all the eigenvalues.

\[
u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n \theta + B_n \sin n \theta)
\]  

(1)

Apply the boundary condition \( u(a, \theta) = f(\theta) \) to determine the coefficients, \( A_0 \), \( A_n \), and \( B_n \).

\[
u(a, \theta) = A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n \theta + B_n \sin n \theta) = f(\theta)
\]

(2)

To find \( A_0 \), integrate both sides of equation (2) with respect to \( \theta \) from 0 to \( 2\pi \).

\[
\int_0^{2\pi} A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n \theta + B_n \sin n \theta) \, d\theta = \int_0^{2\pi} f(\theta) \, d\theta
\]

Split up the integral on the left and bring the constants in front.

\[
A_0 \int_0^{2\pi} d\theta + \sum_{n=1}^{\infty} a^n \left( A_n \int_0^{2\pi} \cos n \theta \, d\theta + B_n \int_0^{2\pi} \sin n \theta \, d\theta \right) = \int_0^{2\pi} f(\theta) \, d\theta
\]

\[
A_0 (2\pi) = \int_0^{2\pi} f(\theta) \, d\theta
\]

So then

\[
A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, d\theta.
\]

(3)
To find $A_n$, multiply both sides of equation (2) by $\cos m\theta$, where $m$ is an integer,

$$A_0 \cos m\theta + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta \cos m\theta + B_n \sin n\theta \cos m\theta) = f(\theta) \cos m\theta$$

and then integrate both sides with respect to $\theta$ from 0 to $2\pi$.

$$\int_{0}^{2\pi} \left[ A_0 \cos m\theta + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta \cos m\theta + B_n \sin n\theta \cos m\theta) \right] d\theta = \int_{0}^{2\pi} f(\theta) \cos m\theta d\theta$$

Split up the integral on the left and bring the constants in front.

$$A_0 \int_{0}^{2\pi} \cos m\theta d\theta + \sum_{n=1}^{\infty} a^n \left( A_n \int_{0}^{2\pi} \cos n\theta \cos m\theta d\theta + B_n \int_{0}^{2\pi} \sin n\theta \cos m\theta d\theta \right) = \int_{0}^{2\pi} f(\theta) \cos m\theta d\theta$$

Because the cosine functions are orthogonal, the second integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$a^n A_n \int_{0}^{2\pi} \cos^2 n\theta d\theta = \int_{0}^{2\pi} f(\theta) \cos n\theta d\theta$$

$$a^n A_n (\pi) = \int_{0}^{2\pi} f(\theta) \cos n\theta d\theta$$

So then

$$A_n = \frac{1}{\pi a^n} \int_{0}^{2\pi} f(\theta) \cos n\theta d\theta. \quad (4)$$

To find $B_n$, multiply both sides of equation (2) by $\sin m\theta$, where $m$ is an integer,

$$A_0 \sin m\theta + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta \sin m\theta + B_n \sin n\theta \sin m\theta) = f(\theta) \sin m\theta$$

and then integrate both sides with respect to $\theta$ from 0 to $2\pi$.

$$\int_{0}^{2\pi} \left[ A_0 \sin m\theta + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta \sin m\theta + B_n \sin n\theta \sin m\theta) \right] d\theta = \int_{0}^{2\pi} f(\theta) \sin m\theta d\theta$$

Split up the integral on the left and bring the constants in front.

$$A_0 \int_{0}^{2\pi} \sin m\theta d\theta + \sum_{n=1}^{\infty} a^n \left( A_n \int_{0}^{2\pi} \cos n\theta \sin m\theta d\theta + B_n \int_{0}^{2\pi} \sin n\theta \sin m\theta d\theta \right) = \int_{0}^{2\pi} f(\theta) \sin m\theta d\theta$$

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Because the sine functions are orthogonal, the third integral on the left is zero if \( n \neq m \). As a result, every term in the infinite series vanishes except for the \( n = m \) one.

\[
a^n B_n \int_0^{2\pi} \sin^2 n\theta \, d\theta = \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta
\]

So then

\[
a^n B_n(\pi) = \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta
\]

Now substitute equations (3), (4), and (5) into equation (1).

\[
u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} f(\bar{\theta}) \, d\bar{\theta} + \sum_{n=1}^{\infty} r^n \left[ \left( \frac{1}{\pi a^n} \int_0^{2\pi} f(\bar{\theta}) \cos n\tilde{\theta} \, d\tilde{\theta} \right) \cos n\theta + \left( \frac{1}{\pi a^n} \int_0^{2\pi} f(\bar{\theta}) \sin n\tilde{\theta} \, d\tilde{\theta} \right) \sin n\theta \right]
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} f(\bar{\theta}) \, d\bar{\theta} + \frac{1}{\pi} \sum_{n=1}^{\infty} a^n \left[ \int_0^{2\pi} f(\bar{\theta}) \cos n\theta \cos n\tilde{\theta} \, d\tilde{\theta} + \int_0^{2\pi} f(\bar{\theta}) \sin n\theta \sin n\tilde{\theta} \, d\tilde{\theta} \right]
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} f(\bar{\theta}) \, d\bar{\theta} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \int_0^{2\pi} f(\bar{\theta}) \cos(n\theta - n\tilde{\theta}) \, d\tilde{\theta}
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[ \frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \operatorname{Re} e^{in(\theta - \tilde{\theta})} \right] \, d\bar{\theta}
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{r}{a} \right)^n \operatorname{Re} e^{in(\theta - \tilde{\theta})} \right] \, d\bar{\theta}
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[ -\frac{1}{2} + \sum_{n=0}^{\infty} \left( \frac{r e^{i(\theta - \tilde{\theta})}}{a} \right)^n \right] \, d\bar{\theta}
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[ -\frac{1}{2} + \operatorname{Re} \left( \frac{1}{1 - \frac{r e^{i(\theta - \tilde{\theta})}}{a}} \right) \right] \, d\bar{\theta}
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[ -\frac{1}{2} + \operatorname{Re} \left( \frac{1}{1 - \frac{r e^{i(\theta - \tilde{\theta})}}{a}} \right) \right] \, d\bar{\theta}
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} f(\bar{\theta}) \left[ -\frac{1}{2} + \operatorname{Re} \left( \frac{1}{1 - \frac{r e^{i(\theta - \tilde{\theta})}}{a}} \right) \right] \, d\bar{\theta}
\]
Continue simplifying the right side.

$$u(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} f(\tilde{\theta}) \left[ -\frac{1}{2} + \frac{1 - \frac{r}{a} \cos(\theta - \tilde{\theta})}{1 - \frac{2r}{a} \cos(\theta - \tilde{\theta}) + \frac{r^2}{a^2}} \right] d\tilde{\theta}$$

$$= \frac{1}{\pi} \int_0^{2\pi} f(\tilde{\theta}) \left[ -1 + \frac{2r}{a} \cos(\theta - \tilde{\theta}) - \frac{r^2}{a^2} + 2 - \frac{2r}{a} \cos(\theta - \tilde{\theta}) \right] \frac{1}{2 \left[ 1 - \frac{2r}{a} \cos(\theta - \tilde{\theta}) + \frac{r^2}{a^2} \right]} d\tilde{\theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\tilde{\theta}) \frac{1 - \frac{r^2}{a^2}}{1 - \frac{2r}{a} \cos(\theta - \tilde{\theta}) + \frac{r^2}{a^2}} d\tilde{\theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(\tilde{\theta}) \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \tilde{\theta}) + r^2} d\tilde{\theta}$$

Therefore,

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\tilde{\theta})}{a^2 - 2ar \cos(\theta - \tilde{\theta}) + r^2} d\tilde{\theta}.$$ 

Note that $f(\theta)$ was defined for $0 \leq \theta \leq 2\pi$. If one defines it for $-\pi \leq \theta \leq \pi$, then the limits of integration will go from $-\pi$ to $\pi$ instead.