Exercise 2.5.8

Solve Laplace's equation inside a circular annulus ($a < r < b$) subject to the boundary conditions [*Hint*: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

(a) $u(a,\theta) = f(\theta), \qquad u(b,\theta) = g(\theta)$

(b) $\frac{\partial u}{\partial r}(a,\theta) = 0, \qquad u(b,\theta) = g(\theta)$

(c) $\frac{\partial u}{\partial r}(a,\theta) = f(\theta), \qquad \frac{\partial u}{\partial r}(b,\theta) = g(\theta)$

If there is a solvability condition, state it and explain it physically.

Solution

Because the Laplace equation is linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form $u(r,\theta) = R(r)\Theta(\theta)$ and plug it into the PDE.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}R(r)\Theta(\theta)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}R(r)\Theta(\theta) = 0$$

$$\frac{\Theta(\theta)}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R(r)}{r^2}\frac{d^2\Theta}{d\theta^2} = 0$$

Multiply both sides by $r^2/[R(r)\Theta(\theta)]$ in order to separate variables.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = 0$$

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2}$$

The only way a function of r can be equal to a function of θ is if both are equal to a constant λ.

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta(\theta)}\frac{d^2\Theta}{d\theta^2} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{aligned} \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) &= \lambda \\ -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Note that it doesn't matter whether the minus sign is grouped with r or θ as long as all eigenvalues are taken into account.

Though it would seem the method of separation of variables would fail because the provided boundary conditions are inhomogeneous, there are two others that aren't listed which are homogeneous.

$$u(r,0) = u(r,2\pi)$$
$$\frac{\partial u}{\partial \theta}(r,0) = \frac{\partial u}{\partial \theta}(r,2\pi)$$

These periodic boundary conditions come from the fact that the annulus is circular. The solution and its slope in the θ-direction must repeat at any angle and this angle plus 2π. Generally, the interval of θ used is the one that $f(\theta)$ and $g(\theta)$ are defined over. Mr. Haberman uses $-\pi < \theta < \pi$ in the textbook, so that's what will be used here. The following boundary conditions will be used instead.

$$u(r,-\pi) = u(r,\pi)$$
$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

Substitute the product solution $u(r,\theta) = R(r)\Theta(\theta)$ into them.

$$u(r,-\pi) = u(r,\pi) \quad \to \quad R(r)\Theta(-\pi) = R(r)\Theta(\pi) \quad \to \quad \Theta(-\pi) = \Theta(\pi)$$
$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \quad \to \quad R(r)\Theta'(-\pi) = R(r)\Theta'(\pi) \quad \to \quad \Theta'(-\pi) = \Theta'(\pi)$$

Now solve the ODE for Θ.

$$\Theta'' = -\lambda\Theta$$

Check to see if there are positive eigenvalues: $\lambda = \mu^2$.

$$\Theta'' = -\mu^2\Theta$$

The general solution can be written in terms of sine and cosine.

$$\Theta(\theta) = C_1 \cos\mu\theta + C_2 \sin\mu\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \mu(-C_1 \sin\mu\theta + C_2\cos\mu\theta)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\Theta(-\pi) = C_1\cos\mu\pi - C_2\sin\mu\pi = C_1\cos\mu\pi + C_2\sin\mu\pi = \Theta(\pi)$$
$$\Theta'(-\pi) = \mu(C_1\sin\mu\pi + C_2\cos\mu\pi) = \mu(-C_1\sin\mu\pi + C_2\cos\mu\pi) = \Theta'(\pi)$$

Cancel the terms common to both sides.

$$-C_2\sin\mu\pi = C_2\sin\mu\pi$$
$$\mu(C_1\sin\mu\pi) = \mu(-C_1\sin\mu\pi)$$

To avoid the trivial solution, we insist that $C_1 \neq 0$ and $C_2 \neq 0$.

$$\sin\mu\pi = 0$$
$$\mu\pi = n\pi, \quad n = 1, 2, \ldots$$
$$\mu = n$$

There are positive eigenvalues $\lambda = n^2$, and the eigenfunctions associated with them are

$$\Theta(\theta) = C_1 \cos \mu\theta + C_2 \sin \mu\theta \quad \rightarrow \quad \Theta_n(\theta) = E \cos n\theta + F \sin n\theta.$$

Using $\lambda = n^2$, solve the ODE for R now.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2$$

Expand the left side.

$$\frac{r}{R}(R' + rR'') = n^2$$

Multiply both sides by R and bring all terms to the left side.

$$r^2R'' + rR' - n^2R = 0$$

This is an equidimensional ODE, so it has solutions of the form $R(r) = r^m$.

$$R = r^m \quad \rightarrow \quad R' = mr^{m-1} \quad \rightarrow \quad R'' = m(m-1)r^{m-2}$$

Substitute these formulas into the ODE and solve the resulting equation for m.

$$r^2m(m-1)r^{m-2} + rmr^{m-1} - 9n^2r^m = 0$$
$$m(m-1)r^m + mr^m - n^2r^m = 0$$
$$m(m-1) + m - n^2 = 0$$
$$m^2 - n^2 = 0$$
$$(m+n)(m-n) = 0$$
$$m = \{-n, n\}$$

Two solutions to the ODE are $R = r^{-n}$ and $R = r^n$. By the principle of superposition, the general solution for R is a linear combination of these two.

$$R(r) = Ar^{-n} + Br^n$$

Now check to see if zero is an eigenvalue: $\lambda = 0$.

$$\Theta'' = 0$$

The general solution is a straight line.

$$\Theta(\theta) = C_3\theta + C_4$$

Apply the boundary conditions here to determine C_3 and C_4.

$$\begin{aligned} \Theta(-\pi) &= -C_3\pi + C_4 = C_3\pi + C_4 = \Theta(\pi) \\ \Theta'(-\pi) &= C_3 = C_3 = \Theta'(\pi) \end{aligned}$$

C_4 cancels from the first equation, leaving $-C_3\pi = C_3\pi$. Only $C_3 = 0$ satisfies this equation, and C_4 remains arbitrary.

$$\Theta(\theta) = C_4$$

This is not the trivial solution, so zero is an eigenvalue. The eigenfunction associated with it is a constant. Now solve the ODE for R with $\lambda = 0$.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Multiply both sides by R/r.

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r\frac{dR}{dr} = D_1$$

Divide both sides by r.

$$\frac{dR}{dr} = \frac{D_1}{r}$$

Integrate both sides with respect to r once more.

$$R(r) = D_1 \ln r + D_2$$

Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$\Theta'' = \gamma^2\Theta$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$\Theta(\theta) = C_5 \cosh\gamma\theta + C_6 \sinh\gamma\theta$$

Differentiate it with respect to θ.

$$\Theta'(\theta) = \gamma(C_5 \sinh\gamma\theta + C_6 \cosh\gamma\theta)$$

Apply the two boundary conditions to determine C_5 and C_6.

$$\begin{aligned}\Theta(-\pi) = C_5\cosh\gamma\pi - C_6\sinh\gamma\pi &= C_5\cosh\gamma\pi + C_6\sinh\gamma\pi = \Theta(\pi)\\ \Theta'(-\pi) = \gamma(-C_5\sinh\gamma\pi + C_6\cosh\gamma\pi) &= \gamma(C_5\sinh\gamma\pi + C_6\cosh\gamma\pi) = \Theta'(\pi)\end{aligned}$$

Cancel the terms common to both sides in each equation.

$$\begin{aligned}-C_6\sinh\gamma\pi &= C_6\sinh\gamma\pi\\ \gamma(-C_5\sinh\gamma\pi) &= \gamma(C_5\sinh\gamma\pi)\end{aligned}$$

There are no nonzero values of γ that can satisfy these equations, which means the only way they are is if $C_5 = 0$ and $C_6 = 0$.

$$\Theta(\theta) = 0$$

The trivial solution is obtained, so there are no negative eigenvalues. According to the principle of superposition, the general solution to the PDE is a linear combination of the eigenfunctions $u = R_n(r)\Theta_n(\theta)$ over all the eigenvalues. Therefore,

$$u(r,\theta) = (A_0\ln r + B_0) + \sum_{n=1}^{\infty}(A_n r^{-n} + B_n r^n)(E_n\cos n\theta + F_n\sin n\theta).$$

The aim now is to use the prescribed boundary conditions in (a), (b), and (c) to determine all of the constants.

Part (a)

Use the given boundary conditions.

$$u(a,\theta) = (A_0 \ln a + B_0) + \sum_{n=1}^{\infty}(A_n a^{-n} + B_n a^n)(E_n \cos n\theta + F_n \sin n\theta) = f(\theta) \tag{1}$$

$$u(b,\theta) = (A_0 \ln b + B_0) + \sum_{n=1}^{\infty}(A_n b^{-n} + B_n b^n)(E_n \cos n\theta + F_n \sin n\theta) = g(\theta) \tag{2}$$

Start by integrating both sides with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi}\left[(A_0 \ln a + B_0) + \sum_{n=1}^{\infty}(A_n a^{-n} + B_n a^n)(E_n \cos n\theta + F_n \sin n\theta)\right] d\theta = \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$\int_{-\pi}^{\pi}\left[(A_0 \ln b + B_0) + \sum_{n=1}^{\infty}(A_n b^{-n} + B_n b^n)(E_n \cos n\theta + F_n \sin n\theta)\right] d\theta = \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

Split up the integrals and bring the constants in front.

$$(A_0 \ln a + B_0)\overbrace{\int_{-\pi}^{\pi} d\theta}^{=\,2\pi} + \sum_{n=1}^{\infty}(A_n a^{-n} + B_n a^n)\left(E_n \overbrace{\int_{-\pi}^{\pi} \cos n\theta\, d\theta}^{=\,0} + F_n \overbrace{\int_{-\pi}^{\pi} \sin n\theta\, d\theta}^{=\,0}\right) = \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$(A_0 \ln b + B_0)\underbrace{\int_{-\pi}^{\pi} d\theta}_{=\,2\pi} + \sum_{n=1}^{\infty}(A_n b^{-n} + B_n b^n)\left(E_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta\, d\theta}_{=\,0} + F_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta\, d\theta}_{=\,0}\right) = \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

Evaluate the integrals.

$$(A_0 \ln a + B_0)(2\pi) = \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$(A_0 \ln b + B_0)(2\pi) = \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

Solving this system of equations for A_0 and B_0 yields

$$\boxed{A_0 = \frac{1}{2\pi \ln \frac{b}{a}} \int_{-\pi}^{\pi} [g(\theta) - f(\theta)]\, d\theta}$$

and

$$\boxed{B_0 = \frac{1}{2\pi \ln \frac{b}{a}} \int_{-\pi}^{\pi} [(\ln b) f(\theta) - (\ln a) g(\theta)]\, d\theta.}$$

Now multiply both sides of equations (1) and (2) by $\cos p\theta$, where p is an integer.

$$(A_0 \ln a + B_0)\cos p\theta + \sum_{n=1}^{\infty}(A_n a^{-n} + B_n a^n)(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta) = f(\theta)\cos p\theta$$

$$(A_0 \ln b + B_0)\cos p\theta + \sum_{n=1}^{\infty}(A_n b^{-n} + B_n b^n)(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta) = g(\theta)\cos p\theta$$

Integrate both sides of each equation with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi}\left[(A_0\ln a+B_0)\cos p\theta+\sum_{n=1}^{\infty}(A_na^{-n}+B_na^n)(E_n\cos n\theta\cos p\theta+F_n\sin n\theta\cos p\theta)\right]d\theta=\int_{-\pi}^{\pi}f(\theta)\cos p\theta\,d\theta$$
$$\int_{-\pi}^{\pi}\left[(A_0\ln b+B_0)\cos p\theta+\sum_{n=1}^{\infty}(A_nb^{-n}+B_nb^n)(E_n\cos n\theta\cos p\theta+F_n\sin n\theta\cos p\theta)\right]d\theta=\int_{-\pi}^{\pi}g(\theta)\cos p\theta\,d\theta$$

Split up the integrals and bring the constants in front.

$$(A_0\ln a+B_0)\overbrace{\int_{-\pi}^{\pi}\cos p\theta\,d\theta}^{=0}+\sum_{n=1}^{\infty}(A_na^{-n}+B_na^n)\left(E_n\int_{-\pi}^{\pi}\cos n\theta\cos p\theta\,d\theta+F_n\overbrace{\int_{-\pi}^{\pi}\sin n\theta\cos p\theta\,d\theta}^{=0}\right)d\theta=\int_{-\pi}^{\pi}f(\theta)\cos p\theta\,d\theta$$
$$(A_0\ln b+B_0)\underbrace{\int_{-\pi}^{\pi}\cos p\theta\,d\theta}_{=0}+\sum_{n=1}^{\infty}(A_nb^{-n}+B_nb^n)\left(E_n\int_{-\pi}^{\pi}\cos n\theta\cos p\theta\,d\theta+F_n\underbrace{\int_{-\pi}^{\pi}\sin n\theta\cos p\theta\,d\theta}_{=0}\right)d\theta=\int_{-\pi}^{\pi}g(\theta)\cos p\theta\,d\theta$$

Because the sine and cosine functions are orthogonal, the third integral in each equation is zero. On the other hand, the second integral is zero if $n\neq p$. Only the $n=p$ term in the sum remains.

$$(A_na^{-n}+B_na^n)\left(E_n\int_{-\pi}^{\pi}\cos^2 n\theta\,d\theta\right)d\theta=\int_{-\pi}^{\pi}f(\theta)\cos n\theta\,d\theta$$
$$(A_nb^{-n}+B_nb^n)\left(E_n\int_{-\pi}^{\pi}\cos^2 n\theta\,d\theta\right)d\theta=\int_{-\pi}^{\pi}g(\theta)\cos n\theta\,d\theta$$

Evaluate the integrals.

$$(A_nE_na^{-n}+B_nE_na^n)(\pi)=\int_{-\pi}^{\pi}f(\theta)\cos n\theta\,d\theta$$
$$(A_nE_nb^{-n}+B_nE_nb^n)(\pi)=\int_{-\pi}^{\pi}g(\theta)\cos n\theta\,d\theta$$

Solving this system of equations for A_nE_n and B_nE_n yields

$$\boxed{A_nE_n = \frac{a^nb^n}{\pi(b^{2n} - a^{2n})} \int_{-\pi}^{\pi} [b^n f(\theta) - a^n g(\theta)] \cos n\theta \, d\theta}$$

and

$$\boxed{B_nE_n = \frac{1}{\pi(b^{2n} - a^{2n})} \int_{-\pi}^{\pi} [b^n g(\theta) - a^n f(\theta)] \cos n\theta \, d\theta.}$$

Now multiply both sides of equations (1) and (2) by $\sin p\theta$, where p is an integer.

$$(A_0 \ln a + B_0) \sin p\theta + \sum_{n=1}^{\infty} (A_n a^{-n} + B_n a^n)(E_n \cos n\theta \sin p\theta + F_n \sin n\theta \sin p\theta) = f(\theta) \sin p\theta$$

$$(A_0 \ln b + B_0) \sin p\theta + \sum_{n=1}^{\infty} (A_n b^{-n} + B_n b^n)(E_n \cos n\theta \sin p\theta + F_n \sin n\theta \sin p\theta) = g(\theta) \sin p\theta$$

Integrate both sides of each equation with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi} \left[(A_0 \ln a + B_0) \sin p\theta + \sum_{n=1}^{\infty} (A_n a^{-n} + B_n a^n)(E_n \cos n\theta \sin p\theta + F_n \sin n\theta \sin p\theta)\right] d\theta = \int_{-\pi}^{\pi} f(\theta) \sin p\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \left[(A_0 \ln b + B_0) \sin p\theta + \sum_{n=1}^{\infty} (A_n b^{-n} + B_n b^n)(E_n \cos n\theta \sin p\theta + F_n \sin n\theta \sin p\theta)\right] d\theta = \int_{-\pi}^{\pi} g(\theta) \sin p\theta \, d\theta$$

Split up the integrals and bring the constants in front.

$$(A_0 \ln a + B_0) \overbrace{\int_{-\pi}^{\pi} \sin p\theta \, d\theta}^{=0} + \sum_{n=1}^{\infty} (A_n a^{-n} + B_n a^n) \left(E_n \overbrace{\int_{-\pi}^{\pi} \cos n\theta \sin p\theta \, d\theta}^{=0} + F_n \int_{-\pi}^{\pi} \sin n\theta \sin p\theta \, d\theta\right) d\theta = \int_{-\pi}^{\pi} f(\theta) \sin p\theta \, d\theta$$

$$(A_0 \ln b + B_0) \underbrace{\int_{-\pi}^{\pi} \sin p\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} (A_n b^{-n} + B_n b^n) \left(E_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta \sin p\theta \, d\theta}_{=0} + F_n \int_{-\pi}^{\pi} \sin n\theta \sin p\theta \, d\theta\right) d\theta = \int_{-\pi}^{\pi} g(\theta) \sin p\theta \, d\theta$$

Because the sine and cosine functions are orthogonal, the second integral in each equation is zero. On the other hand, the third integral is zero if $n \neq p$. Only the $n = p$ term in the sum remains.

$$(A_n a^{-n} + B_n a^n)\left(F_n \int_{-\pi}^{\pi} \sin^2 n\theta \, d\theta\right) d\theta = \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$
$$(A_n b^{-n} + B_n b^n)\left(F_n \int_{-\pi}^{\pi} \sin^2 n\theta \, d\theta\right) d\theta = \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, d\theta$$

Evaluate the integrals.

$$(A_n F_n a^{-n} + B_n F_n a^n)(\pi) = \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$
$$(A_n F_n b^{-n} + B_n F_n b^n)(\pi) = \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, d\theta$$

Solving this system of equations for $A_n F_n$ and $B_n F_n$ yields

$$\boxed{A_n F_n = \frac{a^n b^n}{\pi(b^{2n} - a^{2n})} \int_{-\pi}^{\pi} [b^n f(\theta) - a^n g(\theta)] \sin n\theta \, d\theta}$$

and

$$\boxed{B_n F_n = \frac{1}{\pi(b^{2n} - a^{2n})} \int_{-\pi}^{\pi} [b^n g(\theta) - a^n f(\theta)] \sin n\theta \, d\theta.}$$

With these formulas, it's more convenient to expand the general solution and write it as

$$\boxed{u(r,\theta) = (A_0 \ln r + B_0) + \sum_{n=1}^{\infty} (A_n E_n r^{-n} + B_n E_n r^n) \cos n\theta + \sum_{n=1}^{\infty} (A_n F_n r^{-n} + B_n F_n r^n) \sin n\theta.}$$

Part (b)

Differentiate the general solution with respect to r.

$$\frac{\partial u}{\partial r} = \frac{A_0}{r} + \sum_{n=1}^{\infty} n(-A_n r^{-n-1} + B_n r^{n-1})(E_n \cos n\theta + F_n \sin n\theta).$$

Use the given boundary conditions.

$$\frac{\partial u}{\partial r}(a,\theta) = \frac{A_0}{a} + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})(E_n \cos n\theta + F_n \sin n\theta) = 0 \tag{3}$$

$$u(b,\theta) = (A_0 \ln b + B_0) + \sum_{n=1}^{\infty} (A_n b^{-n} + B_n b^n)(E_n \cos n\theta + F_n \sin n\theta) = g(\theta) \tag{4}$$

Start by integrating both sides with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi} \left[\frac{A_0}{a} + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})(E_n \cos n\theta + F_n \sin n\theta)\right] d\theta = 0$$

$$\int_{-\pi}^{\pi} \left[(A_0 \ln b + B_0) + \sum_{n=1}^{\infty} (A_n b^{-n} + B_n b^n)(E_n \cos n\theta + F_n \sin n\theta)\right] d\theta = \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

Split up the integrals and bring the constants in front.

$$\frac{A_0}{a} \overbrace{\int_{-\pi}^{\pi} d\theta}^{=2\pi} + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})\left(E_n \overbrace{\int_{-\pi}^{\pi} \cos n\theta\, d\theta}^{=0} + F_n \overbrace{\int_{-\pi}^{\pi} \sin n\theta\, d\theta}^{=0}\right) = 0$$

$$(A_0 \ln b + B_0) \underbrace{\int_{-\pi}^{\pi} d\theta}_{=2\pi} + \sum_{n=1}^{\infty} (A_n b^{-n} + B_n b^n)\left(E_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta\, d\theta}_{=0} + F_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta\, d\theta}_{=0}\right) = \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

Evaluate the integrals.

$$\frac{A_0}{a}(2\pi) = 0$$

$$(A_0 \ln b + B_0)(2\pi) = \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

Solving this system of equations yields

$$\boxed{A_0 = 0}$$

and

$$\boxed{B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, d\theta.}$$

Now multiply both sides of equation (3) and equation (4) by $\cos p\theta$, where p is an integer.

$$\frac{A_0}{a}\cos p\theta + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta) = 0$$

$$(A_0 \ln b + B_0)\cos p\theta + \sum_{n=1}^{\infty}(A_n b^{-n} + B_n b^n)(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta) = g(\theta)\cos p\theta$$

Integrate both sides of each equation with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi}\left[\frac{A_0}{a}\cos p\theta + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta)\right] d\theta = 0$$

$$\int_{-\pi}^{\pi}\left[(A_0 \ln b + B_0)\cos p\theta + \sum_{n=1}^{\infty}(A_n b^{-n} + B_n b^n)(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta)\right] d\theta = \int_{-\pi}^{\pi} g(\theta)\cos p\theta\, d\theta$$

Split up the integrals and bring the constants in front.

$$\frac{A_0}{a}\overbrace{\int_{-\pi}^{\pi}\cos p\theta\, d\theta}^{=0} + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})\left(E_n \int_{-\pi}^{\pi}\cos n\theta \cos p\theta\, d\theta + F_n \overbrace{\int_{-\pi}^{\pi}\sin n\theta \cos p\theta\, d\theta}^{=0}\right) = 0$$

$$(A_0 \ln b + B_0)\underbrace{\int_{-\pi}^{\pi}\cos p\theta\, d\theta}_{=0} + \sum_{n=1}^{\infty}(A_n b^{-n} + B_n b^n)\left(E_n \int_{-\pi}^{\pi}\cos n\theta \cos p\theta\, d\theta + F_n \underbrace{\int_{-\pi}^{\pi}\sin n\theta \cos p\theta\, d\theta}_{=0}\right) = \int_{-\pi}^{\pi} g(\theta)\cos p\theta\, d\theta$$

Because the sine and cosine functions are orthogonal, the third integral in each equation is zero. On the other hand, the second integral is zero if $n \neq p$. Only the $n = p$ term in the sum remains.

$$n(-A_n a^{-n-1} + B_n a^{n-1})\left(E_n \int_{-\pi}^{\pi}\cos^2 n\theta\, d\theta\right) = 0$$

$$(A_n b^{-n} + B_n b^n)\left(E_n \int_{-\pi}^{\pi}\cos^2 n\theta\, d\theta\right) = \int_{-\pi}^{\pi} g(\theta)\cos n\theta\, d\theta$$

Evaluate the integrals.

$$n(-A_n E_n a^{-n-1} + B_n E_n a^{n-1})(\pi) = 0$$

$$(A_n E_n b^{-n} + B_n E_n b^n)(\pi) = \int_{-\pi}^{\pi} g(\theta)\cos n\theta\, d\theta$$

Solving this system of equations yields

$$\boxed{A_nE_n = \frac{a^{2n}b^n}{\pi(a^{2n}+b^{2n})}\int_{-\pi}^{\pi} g(\theta)\cos n\theta\,d\theta}$$

and

$$\boxed{B_nE_n = \frac{b^n}{\pi(a^{2n}+b^{2n})}\int_{-\pi}^{\pi} g(\theta)\cos n\theta\,d\theta.}$$

Now multiply both sides of equation (3) and equation (4) by $\sin p\theta$, where p is an integer.

$$\frac{A_0}{a}\sin p\theta + \sum_{n=1}^{\infty} n(-A_na^{-n-1}+B_na^{n-1})(E_n\cos n\theta\sin p\theta + F_n\sin n\theta\sin p\theta) = 0$$

$$(A_0\ln b + B_0)\sin p\theta + \sum_{n=1}^{\infty}(A_nb^{-n}+B_nb^n)(E_n\cos n\theta\sin p\theta + F_n\sin n\theta\sin p\theta) = g(\theta)\sin p\theta$$

Integrate both sides of each equation with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi}\left[\frac{A_0}{a}\sin p\theta + \sum_{n=1}^{\infty} n(-A_na^{-n-1}+B_na^{n-1})(E_n\cos n\theta\sin p\theta + F_n\sin n\theta\sin p\theta)\right]d\theta = 0$$

$$\int_{-\pi}^{\pi}\left[(A_0\ln b + B_0)\sin p\theta + \sum_{n=1}^{\infty}(A_nb^{-n}+B_nb^n)(E_n\cos n\theta\sin p\theta + F_n\sin n\theta\sin p\theta)\right]d\theta = \int_{-\pi}^{\pi} g(\theta)\sin p\theta\,d\theta$$

Split up the integrals and bring the constants in front.

$$\frac{A_0}{a}\overbrace{\int_{-\pi}^{\pi}\sin p\theta\,d\theta}^{=0} + \sum_{n=1}^{\infty} n(-A_na^{-n-1}+B_na^{n-1})\left(E_n\overbrace{\int_{-\pi}^{\pi}\cos n\theta\sin p\theta\,d\theta}^{=0} + F_n\int_{-\pi}^{\pi}\sin n\theta\sin p\theta\,d\theta\right) = 0$$

$$(A_0\ln b + B_0)\underbrace{\int_{-\pi}^{\pi}\sin p\theta\,d\theta}_{=0} + \sum_{n=1}^{\infty}(A_nb^{-n}+B_nb^n)\left(E_n\underbrace{\int_{-\pi}^{\pi}\cos n\theta\sin p\theta\,d\theta}_{=0} + F_n\int_{-\pi}^{\pi}\sin n\theta\sin p\theta\,d\theta\right) = \int_{-\pi}^{\pi} g(\theta)\sin p\theta\,d\theta$$

Because the sine and cosine functions are orthogonal, the second integral in each equation is zero. On the other hand, the third integral is zero if $n \neq p$. Only the $n = p$ term in the sum remains.

$$n(-A_n a^{-n-1} + B_n a^{n-1})\left(F_n \int_{-\pi}^{\pi} \sin^2 n\theta \, d\theta\right) = 0$$

$$(A_n b^{-n} + B_n b^n)\left(F_n \int_{-\pi}^{\pi} \sin^2 n\theta \, d\theta\right) = \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, d\theta$$

Evaluate the integrals.

$$n(-A_n F_n a^{-n-1} + B_n F_n a^{n-1})(\pi) = 0$$

$$(A_n F_n b^{-n} + B_n F_n b^n)(\pi) = \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, d\theta$$

Solving this system of equations yields

$$\boxed{A_n F_n = \frac{a^{2n} b^n}{\pi(a^{2n} + b^{2n})} \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, d\theta}$$

and

$$\boxed{B_n F_n = \frac{b^n}{\pi(a^{2n} + b^{2n})} \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, d\theta.}$$

With these formulas, it's more convenient to expand the general solution and write it as

$$\boxed{u(r,\theta) = (A_0 \ln r + B_0) + \sum_{n=1}^{\infty} (A_n E_n r^{-n} + B_n E_n r^n) \cos n\theta + \sum_{n=1}^{\infty} (A_n F_n r^{-n} + B_n F_n r^n) \sin n\theta.}$$

Part (c)

Differentiate the general solution with respect to r.

$$\frac{\partial u}{\partial r} = \frac{A_0}{r} + \sum_{n=1}^{\infty} n(-A_n r^{-n-1} + B_n r^{n-1})(E_n \cos n\theta + F_n \sin n\theta).$$

Use the given boundary conditions.

$$\frac{\partial u}{\partial r}(a,\theta) = \frac{A_0}{a} + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})(E_n \cos n\theta + F_n \sin n\theta) = f(\theta) \tag{5}$$

$$\frac{\partial u}{\partial r}(b,\theta) = \frac{A_0}{b} + \sum_{n=1}^{\infty} n(-A_n b^{-n-1} + B_n b^{n-1})(E_n \cos n\theta + F_n \sin n\theta) = g(\theta) \tag{6}$$

Start by integrating both sides with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi} \left[\frac{A_0}{a} + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})(E_n \cos n\theta + F_n \sin n\theta)\right] d\theta = \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$\int_{-\pi}^{\pi} \left[\frac{A_0}{b} + \sum_{n=1}^{\infty} n(-A_n b^{-n-1} + B_n b^{n-1})(E_n \cos n\theta + F_n \sin n\theta)\right] d\theta = \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

Split up the integrals and bring the constants in front.

$$\frac{A_0}{a} \overbrace{\int_{-\pi}^{\pi} d\theta}^{=2\pi} + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1}) \left(E_n \overbrace{\int_{-\pi}^{\pi} \cos n\theta\, d\theta}^{=0} + F_n \overbrace{\int_{-\pi}^{\pi} \sin n\theta\, d\theta}^{=0} \right) = \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$\frac{A_0}{b} \underbrace{\int_{-\pi}^{\pi} d\theta}_{=2\pi} + \sum_{n=1}^{\infty} n(-A_n b^{-n-1} + B_n b^{n-1}) \left(E_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta\, d\theta}_{=0} + F_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta\, d\theta}_{=0} \right) = \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

Evaluate the integrals.

$$\frac{A_0}{a}(2\pi) = \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$\frac{A_0}{b}(2\pi) = \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

Therefore,

$$\boxed{A_0 = \frac{a}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{b}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, d\theta.}$$

Now multiply both sides of equation (5) and equation (6) by $\cos p\theta$, where p is an integer.

$$\frac{A_0}{a}\cos p\theta + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta) = f(\theta)\cos p\theta$$

$$\frac{A_0}{b}\cos p\theta + \sum_{n=1}^{\infty} n(-A_n b^{-n-1} + B_n b^{n-1})(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta) = g(\theta)\cos p\theta$$

Integrate both sides of each equation with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi}\left[\frac{A_0}{a}\cos p\theta + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta)\right] d\theta = \int_{-\pi}^{\pi} f(\theta)\cos p\theta \, d\theta$$

$$\int_{-\pi}^{\pi}\left[\frac{A_0}{b}\cos p\theta + \sum_{n=1}^{\infty} n(-A_n b^{-n-1} + B_n b^{n-1})(E_n \cos n\theta \cos p\theta + F_n \sin n\theta \cos p\theta)\right] d\theta = \int_{-\pi}^{\pi} g(\theta)\cos p\theta \, d\theta$$

Split up the integrals and bring the constants in front.

$$\frac{A_0}{a}\overbrace{\int_{-\pi}^{\pi}\cos p\theta \, d\theta}^{=0} + \sum_{n=1}^{\infty} n(-A_n a^{-n-1} + B_n a^{n-1})\left(E_n \int_{-\pi}^{\pi}\cos n\theta \cos p\theta \, d\theta + F_n \overbrace{\int_{-\pi}^{\pi}\sin n\theta \cos p\theta \, d\theta}^{=0}\right) = \int_{-\pi}^{\pi} f(\theta)\cos p\theta \, d\theta$$

$$\frac{A_0}{b}\underbrace{\int_{-\pi}^{\pi}\cos p\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} n(-A_n b^{-n-1} + B_n b^{n-1})\left(E_n \int_{-\pi}^{\pi}\cos n\theta \cos p\theta \, d\theta + F_n \underbrace{\int_{-\pi}^{\pi}\sin n\theta \cos p\theta \, d\theta}_{=0}\right) = \int_{-\pi}^{\pi} g(\theta)\cos p\theta \, d\theta$$

Because the sine and cosine functions are orthogonal, the third integral in each equation is zero. On the other hand, the second integral is zero if $n \neq p$. Only the $n = p$ term in the sum remains.

$$n(-A_n a^{-n-1} + B_n a^{n-1})\left(E_n \int_{-\pi}^{\pi}\cos^2 n\theta \, d\theta\right) = \int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta$$

$$n(-A_n b^{-n-1} + B_n b^{n-1})\left(E_n \int_{-\pi}^{\pi}\cos^2 n\theta \, d\theta\right) = \int_{-\pi}^{\pi} g(\theta)\cos n\theta \, d\theta$$

Evaluate the integrals.

$$n(-A_n E_n a^{-n-1} + B_n E_n a^{n-1})(\pi) = \int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta$$

$$n(-A_n E_n b^{-n-1} + B_n E_n b^{n-1})(\pi) = \int_{-\pi}^{\pi} g(\theta)\cos n\theta \, d\theta$$

Solving this system of equations yields

$$\boxed{A_nE_n = \frac{a^nb^n}{n\pi(b^{2n}-a^{2n})}\int_{-\pi}^{\pi}[a^nbg(\theta) - ab^nf(\theta)]\cos n\theta\,d\theta}$$

and

$$\boxed{B_nE_n = \frac{1}{n\pi(b^{2n}-a^{2n})}\int_{-\pi}^{\pi}[b^{n+1}g(\theta) - a^{n+1}f(\theta)]\cos n\theta\,d\theta.}$$

Now multiply both sides of equation (5) and equation (6) by $\sin p\theta$, where p is an integer.

$$\frac{A_0}{a}\sin p\theta + \sum_{n=1}^{\infty} n(-A_na^{-n-1} + B_na^{n-1})(E_n\cos n\theta\sin p\theta + F_n\sin n\theta\sin p\theta) = f(\theta)\sin p\theta$$

$$\frac{A_0}{b}\sin p\theta + \sum_{n=1}^{\infty} n(-A_nb^{-n-1} + B_nb^{n-1})(E_n\cos n\theta\sin p\theta + F_n\sin n\theta\sin p\theta) = g(\theta)\sin p\theta$$

Integrate both sides of each equation with respect to θ from $-\pi$ to π.

$$\int_{-\pi}^{\pi}\left[\frac{A_0}{a}\sin p\theta + \sum_{n=1}^{\infty} n(-A_na^{-n-1} + B_na^{n-1})(E_n\cos n\theta\sin p\theta + F_n\sin n\theta\sin p\theta)\right]d\theta = \int_{-\pi}^{\pi} f(\theta)\sin p\theta\,d\theta$$

$$\int_{-\pi}^{\pi}\left[\frac{A_0}{b}\sin p\theta + \sum_{n=1}^{\infty} n(-A_nb^{-n-1} + B_nb^{n-1})(E_n\cos n\theta\sin p\theta + F_n\sin n\theta\sin p\theta)\right]d\theta = \int_{-\pi}^{\pi} g(\theta)\sin p\theta\,d\theta$$

Split up the integrals and bring the constants in front.

$$\frac{A_0}{a}\overbrace{\int_{-\pi}^{\pi}\sin p\theta\,d\theta}^{=0} + \sum_{n=1}^{\infty} n(-A_na^{-n-1} + B_na^{n-1})\left(E_n\overbrace{\int_{-\pi}^{\pi}\cos n\theta\sin p\theta\,d\theta}^{=0} + F_n\int_{-\pi}^{\pi}\sin n\theta\sin p\theta\,d\theta\right) = \int_{-\pi}^{\pi} f(\theta)\sin p\theta\,d\theta$$

$$\frac{A_0}{b}\underbrace{\int_{-\pi}^{\pi}\sin p\theta\,d\theta}_{=0} + \sum_{n=1}^{\infty} n(-A_nb^{-n-1} + B_nb^{n-1})\left(E_n\underbrace{\int_{-\pi}^{\pi}\cos n\theta\sin p\theta\,d\theta}_{=0} + F_n\int_{-\pi}^{\pi}\sin n\theta\sin p\theta\,d\theta\right) = \int_{-\pi}^{\pi} g(\theta)\sin p\theta\,d\theta$$

Because the sine and cosine functions are orthogonal, the second integral in each equation is zero. On the other hand, the third integral is zero if $n \neq p$. Only the $n = p$ term in the sum remains.

$$n(-A_n a^{-n-1} + B_n a^{n-1})\left(F_n \int_{-\pi}^{\pi} \sin^2 n\theta \, d\theta\right) = \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

$$n(-A_n b^{-n-1} + B_n b^{n-1})\left(F_n \int_{-\pi}^{\pi} \sin^2 n\theta \, d\theta\right) = \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, d\theta$$

Evaluate the integrals.

$$n(-A_n F_n a^{-n-1} + B_n F_n a^{n-1})(\pi) = \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

$$n(-A_n F_n b^{-n-1} + B_n F_n b^{n-1})(\pi) = \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, d\theta$$

Solving this system of equations yields

$$\boxed{A_n F_n = \frac{a^n b^n}{n\pi(b^{2n} - a^{2n})} \int_{-\pi}^{\pi} [a^n b g(\theta) - a b^n f(\theta)] \sin n\theta \, d\theta}$$

and

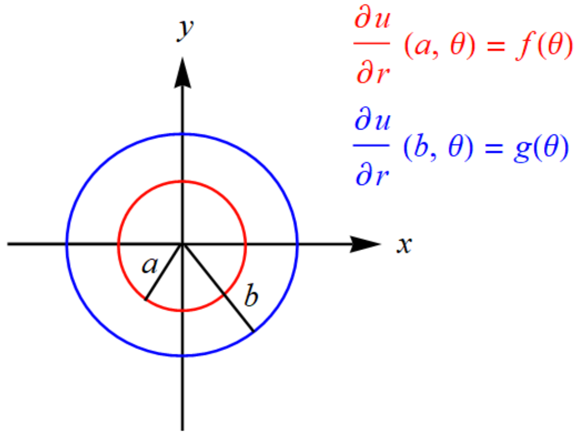
$$\boxed{B_n F_n = \frac{1}{n\pi(b^{2n} - a^{2n})} \int_{-\pi}^{\pi} [b^{n+1} g(\theta) - a^{n+1} f(\theta)] \sin n\theta \, d\theta.}$$

With these formulas, it's more convenient to expand the general solution and write it as

$$\boxed{u(r,\theta) = (A_0 \ln r + B_0) + \sum_{n=1}^{\infty} (A_n E_n r^{-n} + B_n E_n r^n) \cos n\theta + \sum_{n=1}^{\infty} (A_n F_n r^{-n} + B_n F_n r^n) \sin n\theta.}$$

Note that B_0 remains arbitrary.

To obtain the solvability condition, integrate both sides of the PDE over the area of the annulus.

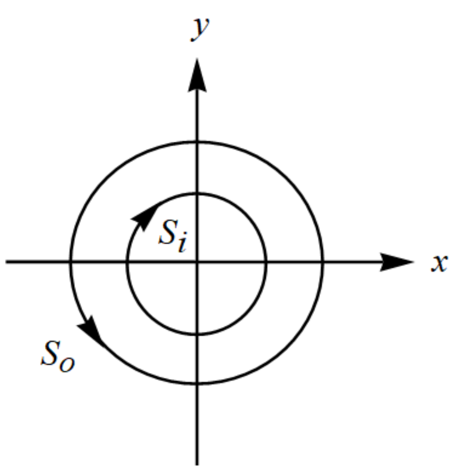


$$\nabla^2 u = 0$$

$$\iint_{\text{annulus}} \nabla^2 u \, dA = \iint_{\text{annulus}} 0 \, dA$$

$$\iint_{\text{annulus}} \nabla \cdot \nabla u \, dA = 0$$

Apply the two-dimensional divergence theorem to turn this double integral into a closed loop integral over the boundary. The path taken around the boundary is such that the annulus area is always to the left.



$$\oint_{\text{annulus bdy}} \nabla u \cdot \hat{\mathbf{n}} \, ds = 0$$

Here $\hat{\mathbf{n}}$ is the outward unit vector normal to the boundary of the annulus. The outward unit vectors are $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ at the inner radius $r = a$ and the outer radius $r = b$, respectively.

$$\int_{S_i} \nabla u \cdot (-\hat{\mathbf{r}}) \, ds + \int_{S_o} \nabla u \cdot \hat{\mathbf{r}} \, ds = 0$$

Evaluate the dot products.

$$\int_{S_i}\left(-\frac{\partial u}{\partial r}\right)ds+\int_{S_o}\left(\frac{\partial u}{\partial r}\right)ds=0$$

Bring the minus sign in front.

$$-\int_{S_i}\frac{\partial u}{\partial r}\,ds+\int_{S_o}\frac{\partial u}{\partial r}\,ds=0$$

Along S_i, u_r is evaluated at $r=a$, and along S_o, u_r is evaluated at $r=b$.

$$-\int_{S_i}\left.\frac{\partial u}{\partial r}\right|_{r=a}ds+\int_{S_o}\left.\frac{\partial u}{\partial r}\right|_{r=b}ds=0$$

The differential of arc length ds is always positive regardless of whether the path around the boundary is clockwise or counterclockwise. So don't mind the orientation when parameterizing the integration paths.

$$-\int_{-\pi}^{\pi}\left.\frac{\partial u}{\partial r}\right|_{r=a}(a\,d\theta)+\int_{-\pi}^{\pi}\left.\frac{\partial u}{\partial r}\right|_{r=b}(b\,d\theta)=0$$

Substitute the prescribed boundary conditions, $u_r(a,\theta)=f(\theta)$ and $u_r(b,\theta)=g(\theta)$.

$$-\int_{-\pi}^{\pi}f(\theta)(a\,d\theta)+\int_{-\pi}^{\pi}g(\theta)(b\,d\theta)=0$$

Therefore, the solvability condition is

$$\boxed{a\int_{-\pi}^{\pi}f(\theta)\,d\theta=b\int_{-\pi}^{\pi}g(\theta)\,d\theta.}$$

Physically, it means the net heat flux into the annulus must be zero for there to be a steady-state temperature.