Exercise 3.2.2

For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$) and determine the Fourier coefficients:

(a) $f(x) = x$

(b) $f(x) = e^{-x}$

(c) $f(x) = \sin \frac{\pi x}{L}$

(d) $f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$

(e) $f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| > L/2 \end{cases}$

(f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$

(g) $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$

Solution

The Fourier series expansion of $f(x)$, which is defined on $-L \le x \le L$ and assumed to be piecewise smooth, is a $2L$-periodic extension of this function over all of x. Where f is continuous, the expansion is given by

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right),$$

and where there are jump discontinuities, the expansion takes the average value of f: $[f(x-) + f(x+)]/2$. The formulas for the coefficients are

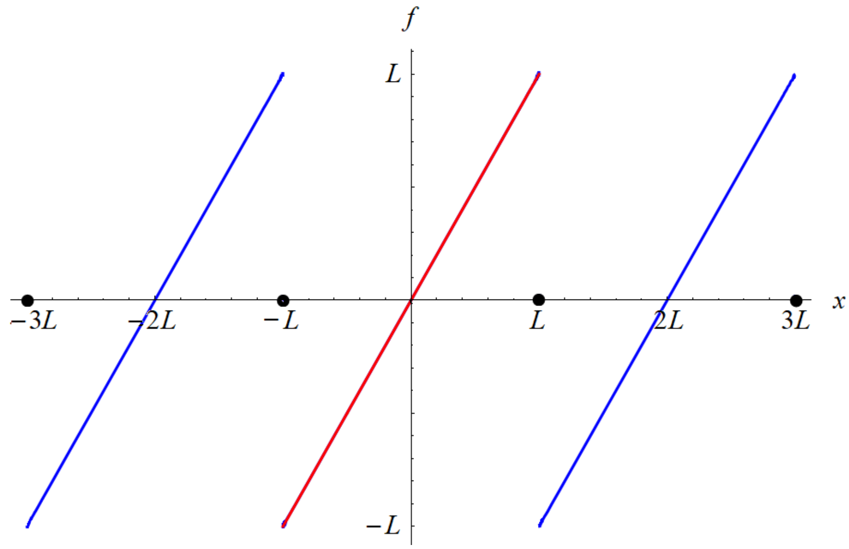
$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$
$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$
$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx,$$

which are obtained by integrating both sides of the Fourier series expansion and taking advantage of the fact that the sine and cosine functions are orthogonal. In the following parts, $f(x)$ will be in red and the expansion will be in blue.

Part (a)

For $f(x) = x$, the coefficients are

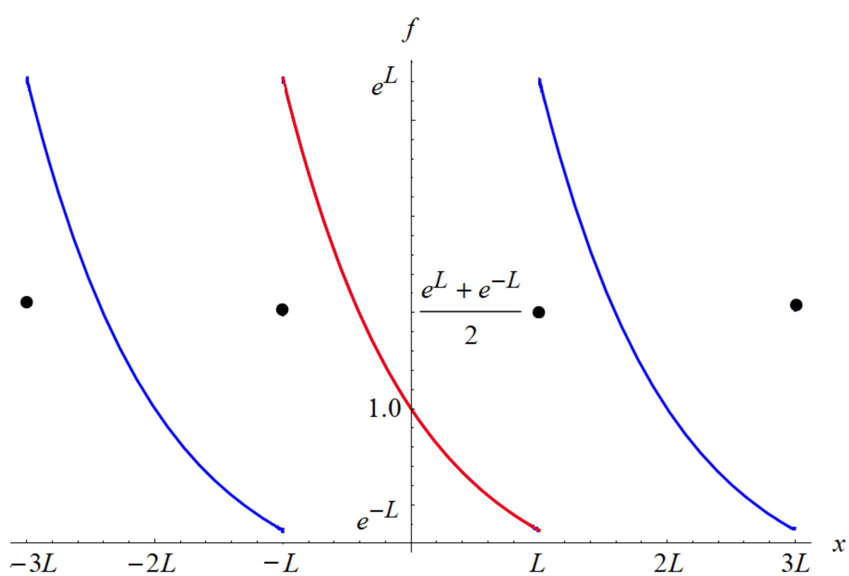
$$A_0 = \frac{1}{2L} \int_{-L}^{L} x\, dx = 0$$
$$A_n = \frac{1}{L} \int_{-L}^{L} x \cos \frac{n\pi x}{L}\, dx = 0$$
$$B_n = \frac{1}{L} \int_{-L}^{L} x \sin \frac{n\pi x}{L}\, dx = -\frac{2(-1)^n L}{n\pi}.$$



Part (b)

For $f(x) = e^{-x}$, the coefficients are

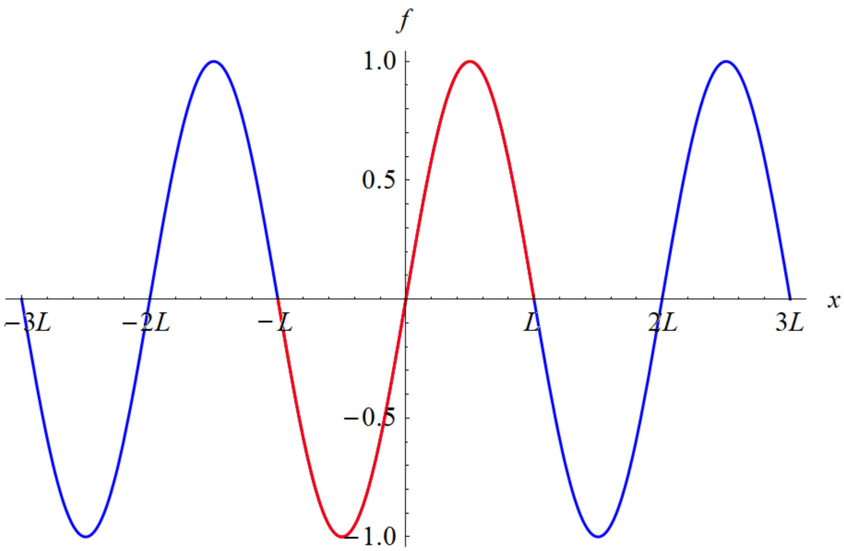
$$A_0 = \frac{1}{2L}\int_{-L}^{L} e^{-x}\,dx = \frac{\sinh L}{L}$$
$$A_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\cos\frac{n\pi x}{L}\,dx = \frac{2(-1)^n L\sinh L}{n^2\pi^2 + L^2}$$
$$B_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\sin\frac{n\pi x}{L}\,dx = \frac{2(-1)^n n\pi\sinh L}{n^2\pi^2 + L^2}.$$



Part (c)

For $f(x) = \sin\frac{\pi x}{L}$, the coefficients are

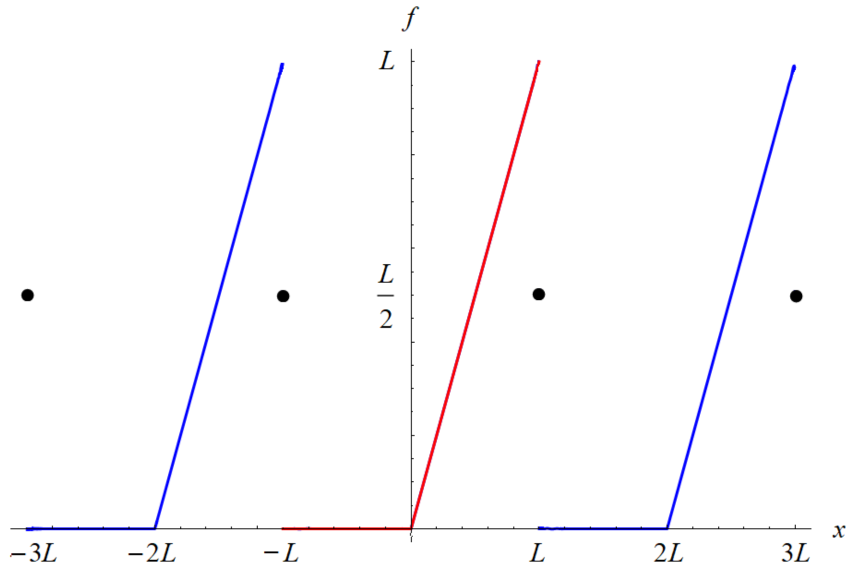
$$A_0 = \frac{1}{2L}\int_{-L}^{L} \sin\frac{\pi x}{L}\,dx = 0$$
$$A_n = \frac{1}{L}\int_{-L}^{L} \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$$
$$B_n = \frac{1}{L}\int_{-L}^{L} \sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n \neq 1 \\ 1 & \text{if } n = 1 \end{cases}.$$



Part (d)

For $f(x) = 0$ if $x < 0$ and $f(x) = x$ if $x > 0$, the coefficients are

$$A_0 = \frac{1}{2L}\left(\int_{-L}^{0} 0\,dx + \int_{0}^{L} x\,dx\right) = \frac{L}{4}$$
$$A_n = \frac{1}{L}\left(\int_{-L}^{0} 0\cos\frac{n\pi x}{L}\,dx + \int_{0}^{L} x\cos\frac{n\pi x}{L}\,dx\right) = \frac{[-1+(-1)^n]L}{n^2\pi^2}$$
$$B_n = \frac{1}{L}\left(\int_{-L}^{0} 0\sin\frac{n\pi x}{L}\,dx + \int_{0}^{L} x\sin\frac{n\pi x}{L}\,dx\right) = -\frac{(-1)^n L}{n\pi}.$$



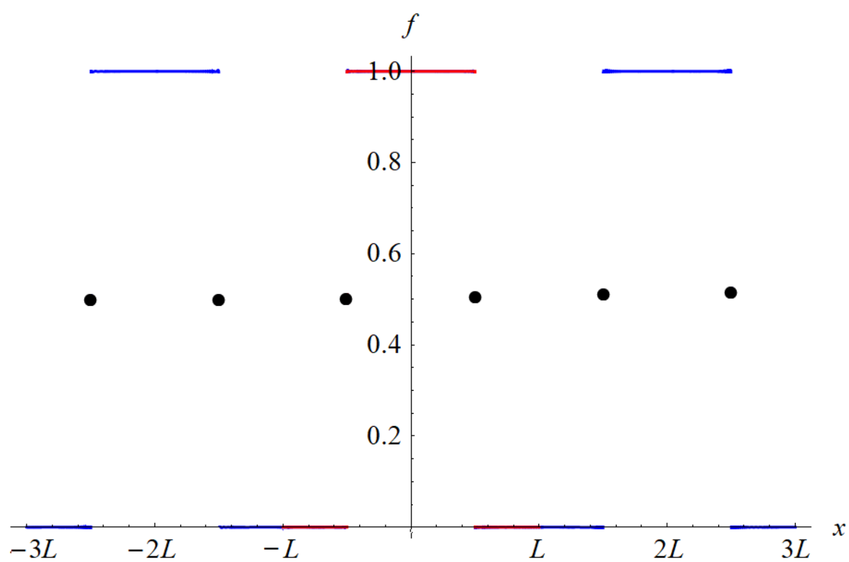
Part (e)

For $f(x) = 1$ if $-L/2 < x < L/2$ and $f(x) = 0$ if $-L < x < -L/2$ and $L/2 < x < L$, the coefficients are

$$A_0 = \frac{1}{2L}\left(\int_{-L}^{-L/2} 0\,dx + \int_{-L/2}^{L/2} dx + \int_{L/2}^{L} 0\,dx\right) = \frac{1}{2}$$

$$A_n = \frac{1}{L}\left(\int_{-L}^{-L/2} 0\cos\frac{n\pi x}{L}\,dx + \int_{-L/2}^{L/2} \cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 0\cos\frac{n\pi x}{L}\,dx\right) = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

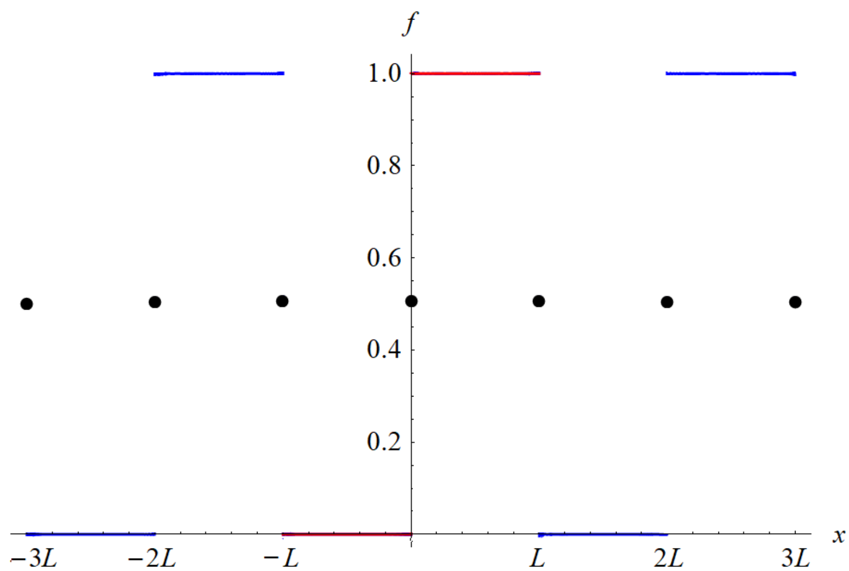
$$B_n = \frac{1}{L}\left(\int_{-L}^{-L/2} 0\sin\frac{n\pi x}{L}\,dx + \int_{-L/2}^{L/2} \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 0\sin\frac{n\pi x}{L}\,dx\right) = 0.$$



Part (f)

For $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x > 0$, the coefficients are

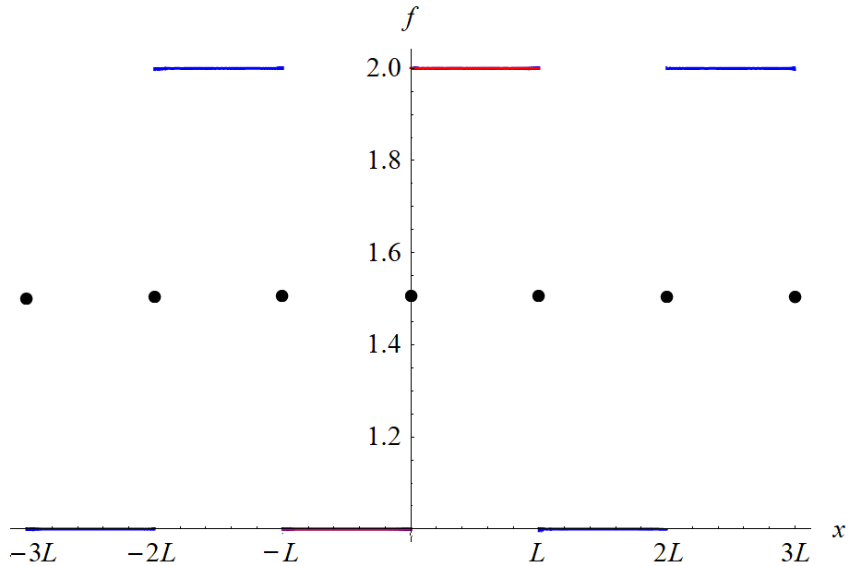
$$A_0 = \frac{1}{2L}\left(\int_{-L}^{0} 0\,dx + \int_0^L dx\right) = \frac{1}{2}$$
$$A_n = \frac{1}{L}\left(\int_{-L}^{0} 0\cos\frac{n\pi x}{L}\,dx + \int_0^L \cos\frac{n\pi x}{L}\,dx\right) = 0$$
$$B_n = \frac{1}{L}\left(\int_{-L}^{0} 0\sin\frac{n\pi x}{L}\,dx + \int_0^L \sin\frac{n\pi x}{L}\,dx\right) = \frac{1-(-1)^n}{n\pi}.$$



Part (g)

For $f(x) = 1$ if $x < 0$ and $f(x) = 2$ if $x > 0$, the coefficients are

$$A_0 = \frac{1}{2L}\left(\int_{-L}^{0} dx + \int_0^L 2\,dx\right) = \frac{3}{2}$$
$$A_n = \frac{1}{L}\left(\int_{-L}^{0} \cos\frac{n\pi x}{L}\,dx + \int_0^L 2\cos\frac{n\pi x}{L}\,dx\right) = 0$$
$$B_n = \frac{1}{L}\left(\int_{-L}^{0} \sin\frac{n\pi x}{L}\,dx + \int_0^L 2\sin\frac{n\pi x}{L}\,dx\right) = \frac{1-(-1)^n}{n\pi}.$$


f
2.0
1.8
1.6
1.4
1.2
x
−3L
−2L
−L
L
2L
3L