Exercise 3.3.12

(a) Graphically show that the even terms (n even) of the Fourier sine series of any function on $0 \leq x \leq L$ are odd (antisymmetric) around $x = L/2$.

(b) Consider a function $f(x)$ that is odd around $x = L/2$. Show that the odd coefficients (n odd) of the Fourier sine series of $f(x)$ on $0 \leq x \leq L$ are zero.

Solution

The Fourier sine series expansion of $f(x)$, a piecewise smooth function defined on $0 \leq x \leq L$, is given by

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

Part (a)

Consider the even terms in the series: $n = 2k$.

$$B_{2k} \sin \frac{2k\pi x}{L}$$

The aim is to show that this function is odd with respect to $x = L/2$, so replace x with $x + L/2$ to translate the sine curve $L/2$ units to the left.

$$B_{2k} \sin \left[\frac{2k\pi}{L}\left(x + \frac{L}{2}\right)\right]$$

$$B_{2k} \sin \left(\frac{2k\pi x}{L} + k\pi\right)$$

$$B_{2k}\left(\sin \frac{2k\pi x}{L} \cos k\pi + \cos \frac{2k\pi x}{L} \sin k\pi\right)$$

$$B_{2k}\left[(-1)^k \sin \frac{2k\pi x}{L} + (0) \cos \frac{2k\pi x}{L}\right]$$

$$B_{2k}(-1)^k \sin \frac{2k\pi x}{L}$$

Swapping x with $-x$ results in the same expression with a minus sign, indicating that this is an odd function.

$$B_{2k}(-1)^k \sin \frac{2k\pi(-x)}{L} \quad \rightarrow \quad -B_{2k}(-1)^k \sin \frac{2k\pi x}{L}$$

Therefore, the even terms in the Fourier sine series expansion are odd with respect to $x = L/2$.

Part (b)

As mentioned before, the Fourier sine coefficients are given by

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Replace x with $x + L/2$ to translate everything to the left by $L/2$ units.

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} f\left(x+\frac{L}{2}\right)\sin\left[\frac{n\pi}{L}\left(x+\frac{L}{2}\right)\right]dx$$

Consider the odd coefficients by setting $n = 2k+1$.

$$\begin{aligned}
B_{2k+1} &= \frac{2}{L}\int_{-L/2}^{L/2} f\left(x+\frac{L}{2}\right)\sin\left[\frac{(2k+1)\pi}{L}\left(x+\frac{L}{2}\right)\right]dx \\
&= \frac{2}{L}\int_{-L/2}^{L/2} f\left(x+\frac{L}{2}\right)\sin\left[\frac{(2k+1)\pi x}{L}+\frac{(2k+1)\pi}{2}\right]dx \\
&= \frac{2}{L}\int_{-L/2}^{L/2} f\left(x+\frac{L}{2}\right)\left[\sin\frac{(2k+1)\pi x}{L}\cos\frac{(2k+1)\pi}{2}+\cos\frac{(2k+1)\pi x}{L}\sin\frac{(2k+1)\pi}{2}\right]dx \\
&= \frac{2}{L}\int_{-L/2}^{L/2} f\left(x+\frac{L}{2}\right)\left[(0)\sin\frac{(2k+1)\pi x}{L}+(-1)^k\cos\frac{(2k+1)\pi x}{L}\right]dx \\
&= \frac{2}{L}\int_{-L/2}^{L/2} f\left(x+\frac{L}{2}\right)(-1)^k\cos\frac{(2k+1)\pi x}{L}\,dx \\
&= \frac{2(-1)^k}{L}\int_{-L/2}^{L/2} f\left(x+\frac{L}{2}\right)\cos\frac{(2k+1)\pi x}{L}\,dx
\end{aligned}$$

Note that f is odd, and cosine is even. The product of an odd function and an even function is odd, and the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned}
B_{2k+1} &= \frac{2(-1)^k}{L}(0) \\
&= 0
\end{aligned}$$

Therefore, the odd coefficients in the Fourier sine series expansion of $f(x)$ are zero if f is odd with respect to $x = L/2$.