Exercise 3.3.15

Consider a function $f(x)$ that is odd around $x = L/2$. Show that the even coefficients (n even) of the Fourier cosine series of $f(x)$ on $0 \leq x \leq L$ are zero.

Solution

The Fourier cosine series expansion of $f(x)$, a piecewise smooth function defined on $0 \leq x \leq L$, is given by

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},$$

where

$$\begin{aligned} A_0 &= \frac{1}{L} \int_0^L f(x)\, dx \\ A_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx. \end{aligned}$$

Replace x with $x + L/2$ to translate everything to the left by $L/2$ units.

$$\begin{aligned} A_0 &= \frac{1}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) dx = 0 \\ A_n &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \cos\left[\frac{n\pi}{L}\left(x + \frac{L}{2}\right)\right] dx. \end{aligned}$$

Consider the even coefficients by setting $n = 2k$.

$$\begin{aligned} A_{2k} &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \cos\left[\frac{2k\pi}{L}\left(x + \frac{L}{2}\right)\right] dx \\ &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \cos\left(\frac{2k\pi x}{L} + k\pi\right) dx \\ &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \left(\cos \frac{2k\pi x}{L} \cos k\pi - \sin \frac{2k\pi x}{L} \sin k\pi\right) dx \\ &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \left[(-1)^k \cos \frac{2k\pi x}{L} - (0) \sin \frac{2k\pi x}{L}\right] dx \\ &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) (-1)^k \cos \frac{2k\pi x}{L}\, dx \\ &= \frac{2(-1)^k}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \cos \frac{2k\pi x}{L}\, dx \end{aligned}$$

Note that f is odd, and cosine is even. The product of an odd function and an even function is odd, and the integral of an odd function over a symmetric interval is zero.

$$A_{2k} = \frac{2(-1)^k}{L}(0) = 0$$

Therefore, A_0 and the even coefficients in the Fourier cosine series expansion of $f(x)$ are zero if f is odd with respect to $x = L/2$.