Exercise 3.4.7

Prove that the Fourier series of a continuous function $u(x,t)$ can be differentiated term by term with respect to the parameter t if $\partial u/\partial t$ is piecewise smooth.

Solution

Since $u(x,t)$ is continuous (on $-L \le x \le L$), it has a Fourier series expansion.

$$u(x,t) = A_0(t) + \sum_{n=1}^{\infty} \left[A_n(t)\cos\frac{n\pi x}{L} + B_n(t)\sin\frac{n\pi x}{L}\right]$$

The coefficients are known to be

$$\begin{aligned} A_0(t) &= \frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx \\ A_n(t) &= \frac{1}{L}\int_{-L}^{L} u(x,t)\cos\frac{n\pi x}{L} \\ B_n(t) &= \frac{1}{L}\int_{-L}^{L} u(x,t)\sin\frac{n\pi x}{L}. \end{aligned}$$

Because $\partial u/\partial t$ is piecewise smooth, it has a Fourier series expansion of its own.

$$\frac{\partial u}{\partial t} = C_0(t) + \sum_{n=1}^{\infty} \left[C_n(t)\cos\frac{n\pi x}{L} + D_n(t)\sin\frac{n\pi x}{L}\right] \tag{1}$$

The aim is to show that

$$C_0(t) = A_0'(t) \quad \text{and} \quad C_n(t) = A_n'(t) \quad \text{and} \quad D_n(t) = B_n'(t).$$

Integrate both sides of equation (1) with respect to x from $-L$ to L.

$$\begin{aligned} \int_{-L}^{L} \frac{\partial u}{\partial t}\,dx &= \int_{-L}^{L} \left\{C_0(t) + \sum_{n=1}^{\infty}\left[C_n(t)\cos\frac{n\pi x}{L} + D_n(t)\sin\frac{n\pi x}{L}\right]\right\} dx \\ &= C_0(t)\int_{-L}^{L} dx + \sum_{n=1}^{\infty}\Bigg[C_n(t)\underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx}_{=\,0} + D_n(t)\underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx}_{=\,0}\Bigg] \\ &= C_0(t)(2L) \end{aligned}$$

Solve for $C_0(t)$.

$$\begin{aligned} C_0(t) &= \frac{1}{2L}\int_{-L}^{L}\frac{\partial u}{\partial t}\,dx \\ &= \frac{d}{dt}\left[\frac{1}{2L}\int_{-L}^{L} u(x,t)\,dx\right] \\ &= A_0'(t) \end{aligned}$$

Multiply both sides of equation (1) by $\cos\frac{p\pi x}{L}$, where p is an integer,

$$\frac{\partial u}{\partial t}\cos\frac{p\pi x}{L} = C_0(t)\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left[C_n(t)\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} + D_n(t)\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right]$$

and then integrate both sides with respect to x from $-L$ to L.

$$\begin{aligned}\int_{-L}^{L}\frac{\partial u}{\partial t}\cos\frac{p\pi x}{L}\,dx &= \int_{-L}^{L}\left\{C_0(t)\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left[C_n(t)\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} + D_n(t)\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right]\right\}dx \\ &= C_0(t)\underbrace{\int_{-L}^{L}\cos\frac{p\pi x}{L}\,dx}_{=0} + \sum_{n=1}^{\infty}\left[C_n(t)\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx + D_n(t)\underbrace{\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx}_{=0}\right]\end{aligned}$$

Because the sine and cosine functions are orthogonal, the third integral is zero for any n and p. Also, the second integral is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$\begin{aligned}\int_{-L}^{L}\frac{\partial u}{\partial t}\cos\frac{n\pi x}{L}\,dx &= C_n(t)\int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx \\ &= C_n(t)(L)\end{aligned}$$

Solve for $C_n(t)$.

$$\begin{aligned}C_n(t) &= \frac{1}{L}\int_{-L}^{L}\frac{\partial u}{\partial t}\cos\frac{n\pi x}{L}\,dx \\ &= \frac{d}{dt}\left[\frac{1}{L}\int_{-L}^{L}u(x,t)\cos\frac{n\pi x}{L}\,dx\right] \\ &= A_n'(t)\end{aligned}$$

Multiply both sides of equation (1) by $\sin\frac{p\pi x}{L}$, where p is an integer,

$$\frac{\partial u}{\partial t}\sin\frac{p\pi x}{L} = C_0(t)\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left[C_n(t)\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + D_n(t)\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\right]$$

and then integrate both sides with respect to x from $-L$ to L.

$$\begin{aligned}\int_{-L}^{L}\frac{\partial u}{\partial t}\sin\frac{p\pi x}{L}\,dx &= \int_{-L}^{L}\left\{C_0(t)\sin\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left[C_n(t)\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L} + D_n(t)\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\right]\right\}dx \\ &= C_0(t)\underbrace{\int_{-L}^{L}\sin\frac{p\pi x}{L}\,dx}_{=0} + \sum_{n=1}^{\infty}\left[C_n(t)\underbrace{\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx}_{=0} + D_n(t)\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx\right]\end{aligned}$$

Because the sine and cosine functions are orthogonal, the second integral is zero for any n and p. Also, the third integral is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$\begin{aligned}\int_{-L}^{L}\frac{\partial u}{\partial t}\sin\frac{n\pi x}{L}\,dx &= D_n(t)\int_{-L}^{L}\sin^2\frac{n\pi x}{L}\,dx \\ &= D_n(t)(L)\end{aligned}$$

Solve for $D_n(t)$.

$$\begin{aligned} D_n(t) &= \frac{1}{L}\int_{-L}^{L} \frac{\partial u}{\partial t} \sin\frac{n\pi x}{L}\, dx \\ &= \frac{d}{dt}\left[\frac{1}{L}\int_{-L}^{L} u(x,t)\sin\frac{n\pi x}{L}\, dx\right] \\ &= B_n'(t) \end{aligned}$$

Therefore, the Fourier series of a continuous function $u(x,t)$ can be differentiated term by term with respect to t if $\partial u/\partial t$ is piecewise smooth.