Exercise 3.5.2

(a) Using (3.3.11) and (3.3.12), obtain the Fourier cosine series of x^2.

(b) From part (a), determine the Fourier sine series of x^3.

Solution

Part (a)

Equation (3.3.11) in the text is the Fourier sine series expansion of x (defined on $0 \leq x \leq L$)

$$x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \tag{3.3.11}$$

and equation (3.3.12) in the text is the formula for the coefficients.

$$B_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L}\, dx = -\frac{2(-1)^n L}{n\pi} \tag{3.3.12}$$

Substitute this formula for the coefficients into the expansion.

$$x = \sum_{n=1}^{\infty} \left[-\frac{2(-1)^n L}{n\pi} \right] \sin \frac{n\pi x}{L}$$

Integrate both sides with respect to x.

$$\begin{aligned}
\frac{x^2}{2} &= \int \sum_{n=1}^{\infty} \left[-\frac{2(-1)^n L}{n\pi} \right] \sin \frac{n\pi x}{L}\, dx + C \\
&= \sum_{n=1}^{\infty} \left[-\frac{2(-1)^n L}{n\pi} \right] \int \sin \frac{n\pi x}{L}\, dx + C \\
&= \sum_{n=1}^{\infty} \left[-\frac{2(-1)^n L}{n\pi} \right] \left(-\frac{L}{n\pi} \right) \cos \frac{n\pi x}{L} + C \\
&= \sum_{n=1}^{\infty} \frac{2(-1)^n L^2}{n^2\pi^2} \cos \frac{n\pi x}{L} + C
\end{aligned}$$

In order to obtain C, integrate both sides with respect to x from 0 to L.

$$\begin{aligned}
\int_0^L \frac{x^2}{2}\, dx &= \int_0^L \left[\sum_{n=1}^{\infty} \frac{2(-1)^n L^2}{n^2\pi^2} \cos \frac{n\pi x}{L} + C \right] dx \\
\frac{L^3}{6} &= \sum_{n=1}^{\infty} \frac{2(-1)^n L^2}{n^2\pi^2} \underbrace{\int_0^L \cos \frac{n\pi x}{L}\, dx}_{=\,0} + C \int_0^L dx \\
&= CL
\end{aligned}$$

Solve for C.

$$C = \frac{L^2}{6}$$

Substitute this into the formula for $x^2/2$.

$$\frac{x^2}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^n L^2}{n^2\pi^2} \cos\frac{n\pi x}{L} + \frac{L^2}{6}$$

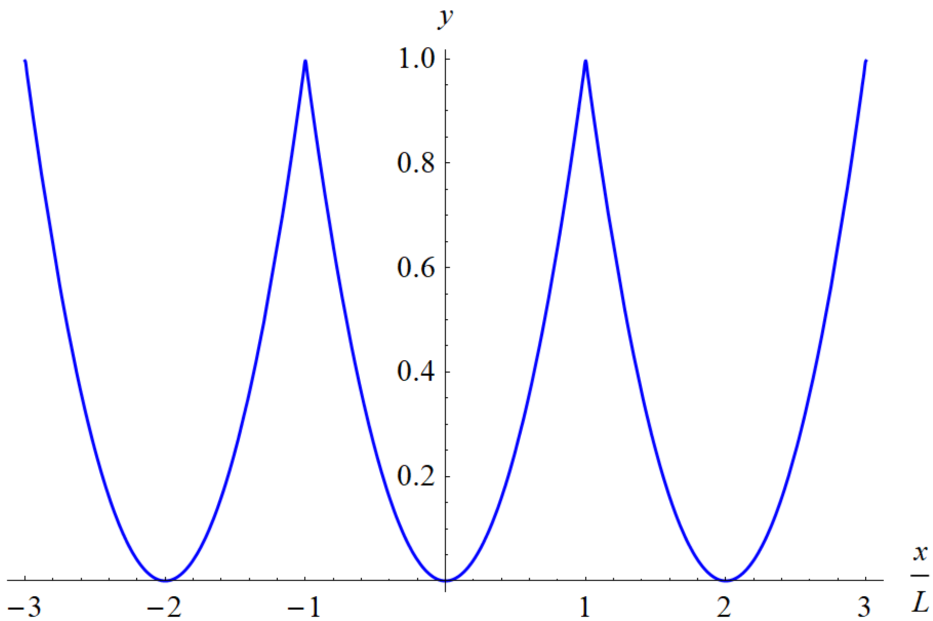
Multiply both sides by 2.

$$x^2 = \sum_{n=1}^{\infty} \frac{4(-1)^n L^2}{n^2\pi^2} \cos\frac{n\pi x}{L} + \frac{L^2}{3}$$

Divide both sides by L^2.

$$\left(\frac{x}{L}\right)^2 = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos\frac{n\pi x}{L} + \frac{1}{3}$$

Below is a graph of the right side versus x/L. Note that the right side is defined on the whole line $(-\infty < x < \infty)$ while the left side is defined only on $0 \leq x \leq L$.



Part (b)

Start with the Fourier cosine series of x^2 in part (a).

$$x^2 = \sum_{n=1}^{\infty} \frac{4(-1)^n L^2}{n^2\pi^2} \cos \frac{n\pi x}{L} + \frac{L^2}{3}$$

Integrate both sides with respect to x.

$$\begin{aligned}
\frac{x^3}{3} &= \int \left[\sum_{n=1}^{\infty} \frac{4(-1)^n L^2}{n^2\pi^2} \cos \frac{n\pi x}{L} + \frac{L^2}{3}\right] dx + D \\
&= \sum_{n=1}^{\infty} \frac{4(-1)^n L^2}{n^2\pi^2} \int \cos \frac{n\pi x}{L}\, dx + \frac{L^2}{3} \int dx + D \\
&= \sum_{n=1}^{\infty} \frac{4(-1)^n L^2}{n^2\pi^2} \left(\frac{L}{n\pi}\right) \sin \frac{n\pi x}{L} + \frac{L^2}{3} x + D
\end{aligned}$$

This equation holds for every value of x, so set $x = 0$ to determine D.

$$0 = D$$

Continue simplifying the formula for $x^3/3$.

$$\begin{aligned}
\frac{x^3}{3} &= \sum_{n=1}^{\infty} \frac{4(-1)^n L^2}{n^2\pi^2} \left(\frac{L}{n\pi}\right) \sin \frac{n\pi x}{L} + \frac{L^2}{3} x \\
&= \sum_{n=1}^{\infty} \frac{4(-1)^n L^3}{n^3\pi^3} \sin \frac{n\pi x}{L} + \frac{L^2}{3} \sum_{n=1}^{\infty} \left[-\frac{2(-1)^n L}{n\pi}\right] \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \frac{4(-1)^n L^3}{n^3\pi^3} \sin \frac{n\pi x}{L} - \sum_{n=1}^{\infty} \frac{2(-1)^n L^3}{3n\pi} \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left[\frac{4(-1)^n L^3}{n^3\pi^3} - \frac{2(-1)^n L^3}{3n\pi}\right] \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left(\frac{4L^3}{n^3\pi^3} - \frac{2L^3}{3n\pi}\right) (-1)^n \sin \frac{n\pi x}{L}
\end{aligned}$$

Multiply both sides by 3.

$$x^3 = \sum_{n=1}^{\infty} \left(\frac{12L^3}{n^3\pi^3} - \frac{2L^3}{n\pi}\right) (-1)^n \sin \frac{n\pi x}{L}$$

Divide both sides by L^3.

$$\left(\frac{x}{L}\right)^3 = \sum_{n=1}^{\infty} \left(\frac{12}{n^3\pi^3} - \frac{2}{n\pi}\right) (-1)^n \sin \frac{n\pi x}{L}$$

Below is a graph of the right side versus x/L. Note that the right side is defined on the whole line $(-\infty < x < \infty)$ while the left side is defined only on $0 \leq x \leq L$.