Exercise 3.3.13

Consider a function $f(x)$ that is even around $x = L/2$. Show that the even coefficients (n even) of the Fourier sine series of $f(x)$ on $0 \le x \le L$ are zero.

Solution

The Fourier sine series expansion of $f(x)$, a piecewise smooth function defined on $0 \le x \le L$, is given by

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$$

Replace x with $x + L/2$ to translate everything to the left by $L/2$ units.

$$B_n = \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin\left[\frac{n\pi}{L}\left(x + \frac{L}{2}\right)\right] dx$$

Consider the even coefficients by setting $n = 2k$.

$$\begin{aligned}
B_{2k} &= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin\left[\frac{2k\pi}{L}\left(x + \frac{L}{2}\right)\right] dx \\
&= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin\left(\frac{2k\pi x}{L} + k\pi\right) dx \\
&= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \left(\sin \frac{2k\pi x}{L} \cos k\pi + \cos \frac{2k\pi x}{L} \sin k\pi\right) dx \\
&= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \left[(-1)^k \sin \frac{2k\pi x}{L} + (0) \cos \frac{2k\pi x}{L}\right] dx \\
&= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) (-1)^k \sin \frac{2k\pi x}{L}\, dx \\
&= \frac{2(-1)^k}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin \frac{2k\pi x}{L}\, dx
\end{aligned}$$

Note that f is even, and sine is odd. The product of an even function and an odd function is odd, and the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned}
B_{2k} &= \frac{2(-1)^k}{L}(0) \\
&= 0
\end{aligned}$$

Therefore, the even coefficients in the Fourier sine series expansion of $f(x)$ are zero if f is even with respect to $x = L/2$.