Exercise 3.3.13

Consider a function \( f(x) \) that is even around \( x = L/2 \). Show that the even coefficients \( (n \text{ even}) \) of the Fourier sine series of \( f(x) \) on \( 0 \leq x \leq L \) are zero.

Solution

The Fourier sine series expansion of \( f(x) \), a piecewise smooth function defined on \( 0 \leq x \leq L \), is given by

\[
f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},
\]

where

\[
B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.
\]

Replace \( x \) with \( x + L/2 \) to translate everything to the left by \( L/2 \) units.

\[
B_n = \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin \left[ \frac{n\pi}{L}\left(x + \frac{L}{2}\right) \right] \, dx
\]

Consider the even coefficients by setting \( n = 2k \).

\[
B_{2k} = \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin \left[ \frac{2k\pi}{L}\left(x + \frac{L}{2}\right) \right] \, dx
\]

\[
= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin \left( \frac{2k\pi x}{L} + k\pi \right) \, dx
\]

\[
= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \left( \sin \frac{2k\pi x}{L} \cos k\pi + \cos \frac{2k\pi x}{L} \sin k\pi \right) \, dx
\]

\[
= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \left[ (-1)^k \sin \frac{2k\pi x}{L} + (0) \cos \frac{2k\pi x}{L} \right] \, dx
\]

\[
= \frac{2}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) (-1)^k \sin \frac{2k\pi x}{L} \, dx
\]

\[
= \frac{2(-1)^k}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin \frac{2k\pi x}{L} \, dx
\]

Note that \( f \) is even, and sine is odd. The product of an even function and an odd function is odd, and the integral of an odd function over a symmetric interval is zero.

\[
B_{2k} = \frac{2(-1)^k}{L} \int_{-L/2}^{L/2} f\left(x + \frac{L}{2}\right) \sin \frac{2k\pi x}{L} \, dx
\]

\[
= 0
\]

Therefore, the even coefficients in the Fourier sine series expansion of \( f(x) \) are zero if \( f \) is even with respect to \( x = L/2 \).