Exercise 3.3.14

(a) Consider a function \( f(x) \) that is even around \( x = L/2 \). Show that the odd coefficients \( (n \text{ odd}) \) of the Fourier cosine series of \( f(x) \) on \( 0 \leq x \leq L \) are zero.

(b) Explain the result of part (a) by considering a Fourier cosine series of \( f(x) \) on the interval \( 0 \leq x \leq L/2 \).

Solution

The Fourier cosine series expansion of \( f(x) \), a piecewise smooth function defined on \( 0 \leq x \leq L \), is given by

\[
f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},
\]

where

\[
A_0 = \frac{1}{L} \int_0^L f(x) \, dx
\]

\[
A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx.
\]

Replace \( x \) with \( x + L/2 \) to translate everything to the left by \( L/2 \) units.

\[
A_0 = \frac{1}{L} \int_{-L/2}^{L/2} f \left( x + \frac{L}{2} \right) \, dx
\]

\[
A_n = \frac{2}{L} \int_{-L/2}^{L/2} f \left( x + \frac{L}{2} \right) \cos \left[ \frac{n\pi}{L} \left( x + \frac{L}{2} \right) \right] \, dx.
\]

Consider the odd coefficients by setting \( n = 2k + 1 \).

\[
A_{2k+1} = \frac{2}{L} \int_{-L/2}^{L/2} f \left( x + \frac{L}{2} \right) \cos \left[ \frac{(2k+1)\pi}{L} \left( x + \frac{L}{2} \right) \right] \, dx
\]

\[
= \frac{2}{L} \int_{-L/2}^{L/2} f \left( x + \frac{L}{2} \right) \cos \left[ \frac{(2k+1)\pi x}{L} + \frac{(2k+1)\pi}{2} \right] \, dx
\]

\[
= \frac{2}{L} \int_{-L/2}^{L/2} f \left( x + \frac{L}{2} \right) \left[ \cos \left( \frac{(2k+1)\pi x}{L} \right) \cos \left( \frac{(2k+1)\pi}{2} \right) - \sin \left( \frac{(2k+1)\pi x}{L} \right) \sin \left( \frac{(2k+1)\pi}{2} \right) \right] \, dx
\]

\[
= \frac{2}{L} \int_{-L/2}^{L/2} f \left( x + \frac{L}{2} \right) \left[ (0) \cos \left( \frac{(2k+1)\pi x}{L} \right) - (-1)^k \sin \left( \frac{(2k+1)\pi x}{L} \right) \right] \, dx
\]

\[
= \frac{2}{L} \int_{-L/2}^{L/2} f \left( x + \frac{L}{2} \right) (-1)^{k+1} \sin \left( \frac{(2k+1)\pi x}{L} \right) \, dx
\]

\[
= \frac{2(-1)^{k+1}}{L} \int_{-L/2}^{L/2} f \left( x + \frac{L}{2} \right) \sin \left( \frac{(2k+1)\pi x}{L} \right) \, dx
\]

Note that \( f \) is even, and sine is odd. The product of an even function and an odd function is odd, and the integral of an odd function over a symmetric interval is zero.

\[
A_{2k+1} = \frac{2(-1)^{k+1}}{L}(0) = 0
\]

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Therefore, the odd coefficients in the Fourier cosine series expansion of $f(x)$ are zero if $f$ is even with respect to $x = L/2$. As an example, consider the Fourier cosine series of $f(x)$ on the interval $0 \leq x \leq L/2$.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{2n\pi x}{L}$$

This series is the $L$-periodic even extension of $f(x)$ to the whole line ($-\infty < x < \infty$), so it is even with respect to $x = L/2$. The point to note is that the series has no cosine terms with odd multiples of $\pi x/L$. 

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