Exercise 3.3.16

Fourier series can be defined on other intervals besides $-L \leq x \leq L$. Suppose that $g(y)$ is defined for $a \leq y \leq b$. Represent $g(y)$ using periodic trigonometric functions with period $b-a$. Determine formulas for the coefficients. [*Hint*: Use the linear transformation

$$y = \frac{a+b}{2} + \frac{b-a}{2L}x.]$$

Solution

Observe that plugging in $-L$ for x in this linear transformation gives a and that plugging in L for x gives b. A piecewise smooth function $f(x)$ that is defined on $-L \leq x \leq L$ has a Fourier series expansion given by

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L},$$

where

$$\begin{aligned}
A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.
\end{aligned}$$

Solve the linear transformation for x,

$$x = \frac{L(2y-b-a)}{b-a},$$

and substitute it into the Fourier series.

$$f(x(y)) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left[\frac{n\pi}{L}\frac{L(2y-b-a)}{b-a}\right] + \sum_{n=1}^{\infty} B_n \sin\left[\frac{n\pi}{L}\frac{L(2y-b-a)}{b-a}\right]$$

$$\boxed{g(y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi(2y-b-a)}{b-a} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi(2y-b-a)}{b-a}}$$

This piecewise smooth function $g(y)$ is defined on $a \leq y \leq b$, and its series has period $b-a$. Now make the substitution in the integral formulas for the coefficients. Note that $dx = 2L/(b-a)\,dy$.

$$A_0 = \frac{1}{2L}\int_a^b f(x(y))\,\frac{2L}{b-a}\,dy \quad \rightarrow \quad \boxed{A_0 = \frac{1}{b-a}\int_a^b g(y)\,dy}$$

$$A_n = \frac{1}{L}\int_a^b f(x(y))\cos\left[\frac{n\pi}{L}\frac{L(2y-b-a)}{b-a}\right]\frac{2L}{b-a}\,dy \quad \rightarrow \quad \boxed{A_n = \frac{2}{b-a}\int_a^b g(y)\cos\frac{n\pi(2y-b-a)}{b-a}\,dy}$$

$$B_n = \frac{1}{L}\int_a^b f(x(y))\sin\left[\frac{n\pi}{L}\frac{L(2y-b-a)}{b-a}\right]\frac{2L}{b-a}\,dy \quad \rightarrow \quad \boxed{B_n = \frac{2}{b-a}\int_a^b g(y)\sin\frac{n\pi(2y-b-a)}{b-a}\,dy}$$