Exercise 3.3.17

Consider

$$\int_0^1 \frac{dx}{x^2+1}.$$

(a) Evaluate explicitly.

(b) Use the Taylor series of $1/(1+x^2)$ (itself a geometric series) to obtain an infinite series for the integral.

(c) Equate part (a) to part (b) in order to derive a formula for π.

Solution

Make the trigonometric substitution,

$$\begin{aligned} x &= \tan\theta \\ dx &= \sec^2\theta\, d\theta. \end{aligned}$$

The integral then becomes

$$\begin{aligned} \int_0^1 \frac{dx}{x^2+1} &= \int_{\tan^{-1}(0)}^{\tan^{-1}(1)} \frac{\sec^2\theta\, d\theta}{(\tan\theta)^2+1} \\ &= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^2\theta} \\ &= \int_0^{\pi/4} d\theta \\ &= \frac{\pi}{4}. \end{aligned}$$

Since the integrand can be written as an infinite series,

$$\frac{1}{x^2+1} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

the integral can also be written as

$$\begin{aligned} \int_0^1 \frac{dx}{x^2+1} &= \int_0^1 \sum_{n=0}^{\infty}(-1)^n x^{2n}\, dx \\ &= \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{2n}\, dx \\ &= \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{2n+1}\right). \end{aligned}$$

Therefore,

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$