Exercise 3.3.2

For the following functions, sketch the Fourier sine series of \( f(x) \) and determine its Fourier coefficients:

(a) \( f(x) = \cos \frac{\pi x}{L} \) [Verify formula (3.3.13).]

(b) \( f(x) = \begin{cases} 
1 & x < L/6 \\
3 & L/6 < x < L/2 \\
0 & x > L/2 
\end{cases} \)

(c) \( f(x) = \begin{cases} 
0 & x < L/2 \\
x & x > L/2 
\end{cases} \)

(d) \( f(x) = \begin{cases} 
1 & x < L/2 \\
0 & x > L/2 
\end{cases} \)

Solution

Assume that \( f(x) \) is a piecewise smooth function on the interval \( 0 \leq x \leq L \). The odd extension of \( f(x) \) to the whole line with period \( 2L \) is given by the Fourier sine series expansion,

\[
f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{L},
\]

at points where \( f(x) \) is continuous and by the average of the left-hand and right-hand limits at points of discontinuity. The coefficients \( B_n \) are obtained by multiplying both sides by \( \sin \frac{p \pi x}{L} \) (\( p \) being an integer), integrating both sides with respect to \( x \) from 0 to \( L \), and taking advantage of the fact that sine functions are orthogonal with one another.

\[
B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx
\]

Part (a)

For \( f(x) = \cos \frac{\pi x}{L} \), the coefficients are

\[
B_n = \frac{2}{L} \int_0^L \cos \frac{\pi x}{L} \sin \frac{n \pi x}{L} \, dx
\]

\[
= \frac{2}{L} \int_0^L \frac{1}{2} \left[ \sin \left( \frac{\pi x}{L} + \frac{n \pi x}{L} \right) - \sin \left( \frac{\pi x}{L} - \frac{n \pi x}{L} \right) \right] \, dx
\]

\[
= \frac{1}{L} \left[ \int_0^L \sin \left( \frac{1+n \pi}{L} x \right) \, dx - \int_0^L \sin \left( \frac{1-n \pi}{L} x \right) \, dx \right] = 0 \quad \text{if } n = 1
\]

\[
= \frac{1}{L} \left[ \frac{(1+n) \pi}{(1+n) \pi} - \frac{(1-n) \pi}{(1-n) \pi} \right] = \begin{cases} 
0 & n = 1 \\
2[1 + (-1)^n] & n \neq 1
\end{cases}
\]

\[
= \begin{cases} 
0 & n \text{ odd} \\
\frac{4n}{(n^2 - 1) \pi} & n \text{ even}
\end{cases}
\]

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Part (b)

For \( f(x) = 1 \) if \( x < L/6 \) and \( f(x) = 3 \) if \( L/6 < x < L/2 \) and \( f(x) = 0 \) if \( x > L/2 \), the coefficients are

\[
B_n = \frac{2}{L} \left( \int_0^{L/6} \sin \frac{n\pi x}{L} \, dx + \int_{L/6}^{L/2} 3 \sin \frac{n\pi x}{L} \, dx + \int_{L/2}^L 0 \sin \frac{n\pi x}{L} \, dx \right)
= \frac{2}{n\pi} \left( 1 + 2 \cos \frac{n\pi}{6} - 3 \cos \frac{n\pi}{2} \right).
\]
Part (c)

For $f(x) = 0$ if $x < L/2$ and $f(x) = x$ if $x > L/2$, the coefficients are

$$B_n = \frac{2}{L} \left( \int_0^{L/2} 0 \sin \frac{n\pi x}{L} \, dx + \int_{L/2}^{L} x \sin \frac{n\pi x}{L} \, dx \right) = \frac{L}{n^2 \pi^2} \left\{ n \pi \cos \frac{n\pi}{2} - 2 \left[ (-1)^n n \pi + \sin \frac{n\pi}{2} \right] \right\}.$$ 

Part (d)

For $f(x) = 1$ if $x < L/2$ and $f(x) = 0$ if $x > L/2$, the coefficients are

$$B_n = \frac{2}{L} \left( \int_0^{L/2} \sin \frac{n\pi x}{L} \, dx + \int_{L/2}^{L} 0 \sin \frac{n\pi x}{L} \, dx \right) = \frac{4}{n\pi} \sin^2 \frac{n\pi}{4}.$$