Exercise 3.3.4

Sketch the Fourier cosine series of $f(x) = \sin \pi x/L$. Briefly discuss.

Solution

Assume that $f(x)$ is a piecewise smooth function on the interval $0 \leq x \leq L$. The even extension of $f(x)$ to the whole line with period $2L$ is given by the Fourier cosine series expansion,

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},$$

at points where $f(x)$ is continuous and by the average of the left-hand and right-hand limits at points of discontinuity. The coefficients A_n are obtained by multiplying both sides by $\cos \frac{p\pi x}{L}$ (p being an integer), integrating both sides with respect to x from 0 to L, and taking advantage of the fact that cosine functions are orthogonal with one another.

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

A_0 is obtained just by integrating both sides of the series expansion with respect to x from 0 to L.

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

For $f(x) = \sin \pi x/L$ in particular, we have

$$A_0 = \frac{1}{L} \int_0^L \sin \frac{\pi x}{L}\, dx = \frac{2}{\pi}$$

and

$$\begin{aligned}
A_n &= \frac{2}{L} \int_0^L \sin \frac{\pi x}{L} \cos \frac{n\pi x}{L}\, dx \\
&= \frac{2}{L} \int_0^L \frac{1}{2} \left[\sin\left(\frac{\pi x}{L} + \frac{n\pi x}{L}\right) + \sin\left(\frac{\pi x}{L} - \frac{n\pi x}{L}\right) \right] dx \\
&= \frac{2}{L} \int_0^L \frac{1}{2} \left[\sin \frac{(1+n)\pi x}{L} + \sin \frac{(1-n)\pi x}{L} \right] dx \\
&= \frac{1}{L} \left[\int_0^L \sin \frac{(1+n)\pi x}{L}\, dx + \int_0^L \sin \frac{(1-n)\pi x}{L}\, dx \right] = 0 \quad \text{if } n = 1 \\
&= \frac{1}{L} \left[\frac{[1 + (-1)^n]L}{(1+n)\pi} + \frac{[1 + (-1)^n]L}{(1-n)\pi} \right] \quad \text{if } n \neq 1 \\
&= \begin{cases} 0 & n = 1 \\ -\dfrac{2[1 + (-1)^n]}{(n^2 - 1)\pi} & n \neq 1 \end{cases} \\
&= \begin{cases} 0 & n \text{ odd} \\ -\dfrac{4}{(n^2 - 1)\pi} & n \text{ even} \end{cases}.
\end{aligned}$$

Below is a plot of $f(x) = \sin \frac{\pi x}{L}$ and its even extension to the whole line.