Exercise 3.3.5

For the following functions, sketch the Fourier cosine series of \( f(x) \) and determine its Fourier coefficients:

(a) \( f(x) = x^2 \)

(b) \( f(x) = \begin{cases} 
1 & x < L/6 \\
3 & L/6 < x < L/2 \\
0 & x > L/2
\end{cases} \)

(c) \( f(x) = \begin{cases} 
0 & x < L/2 \\
x & x > L/2
\end{cases} \)

Solution

Assume that \( f(x) \) is a piecewise smooth function on the interval \( 0 \leq x \leq L \). The even extension of \( f(x) \) to the whole line with period \( 2L \) is given by the Fourier cosine series expansion,

\[
f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},
\]

at points where \( f(x) \) is continuous and by the average of the left-hand and right-hand limits at points of discontinuity. The coefficients \( A_n \) are obtained by multiplying both sides by \( \cos \frac{p\pi x}{L} \) (\( p \) being an integer), integrating both sides with respect to \( x \) from 0 to \( L \), and taking advantage of the fact that cosine functions are orthogonal with one another.

\[
A_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \,dx
\]

\( A_0 \) is obtained just by integrating both sides of the series expansion with respect to \( x \) from 0 to \( L \).

\[
A_0 = \frac{1}{L} \int_{0}^{L} f(x) \,dx
\]

Part (a)

For \( f(x) = x^2 \), the coefficients are

\[
A_0 = \frac{1}{L} \int_{0}^{L} x^2 \,dx = \frac{L^2}{3}
\]

and

\[
A_n = \frac{2}{L} \int_{0}^{L} x^2 \cos \frac{n\pi x}{L} \,dx = \frac{4(-1)^n L^2}{n^2\pi^2}.
\]
Part (b)

For \( f(x) = 1 \) if \( x < L/6 \) and \( f(x) = 3 \) if \( L/6 < x < L/2 \) and \( f(x) = 0 \) if \( x > L/2 \), the coefficients are

\[
A_0 = \frac{1}{L} \left( \int_0^{L/6} 1 \, dx + \int_{L/6}^{L/2} 3 \, dx + \int_{L/2}^L 0 \, dx \right) = \frac{7}{6}
\]

and

\[
A_n = \frac{2}{L} \left( \int_0^{L/6} \cos \frac{n\pi x}{L} \, dx + \int_{L/6}^{L/2} 3 \cos \frac{n\pi x}{L} \, dx + \int_{L/2}^L 0 \cos \frac{n\pi x}{L} \, dx \right) = \frac{2}{n\pi} \left( 3 \sin \frac{n\pi}{2} - 2 \sin \frac{n\pi}{6} \right).
\]

Below is a graph of the function and its even extension to the whole line.

![Graph of the function and its even extension to the whole line.](image-url)
Part (c)

For \( f(x) = 0 \) if \( x < \frac{L}{2} \) and \( f(x) = x \) if \( x > \frac{L}{2} \), the coefficients are

\[
A_0 = \frac{1}{L} \left( \int_0^{L/2} 0 \, dx + \int_{L/2}^L x \, dx \right) = \frac{3L}{8}
\]

and

\[
A_n = \frac{2}{L} \left( \int_0^{L/2} 0 \cos \frac{n\pi x}{L} \, dx + \int_{L/2}^L x \cos \frac{n\pi x}{L} \, dx \right) = \frac{L}{n^2 \pi^2} \left[ 2(-1)^n - 2 \cos \frac{n\pi}{2} - n\pi \sin \frac{n\pi}{2} \right].
\]

Below is a graph of the function and its even extension to the whole line.