Exercise 3.3.5

For the following functions, sketch the Fourier cosine series of $f(x)$ and determine its Fourier coefficients:

(a) $f(x) = x^2$

(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

Solution

Assume that $f(x)$ is a piecewise smooth function on the interval $0 \leq x \leq L$. The even extension of $f(x)$ to the whole line with period $2L$ is given by the Fourier cosine series expansion,

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},$$

at points where $f(x)$ is continuous and by the average of the left-hand and right-hand limits at points of discontinuity. The coefficients A_n are obtained by multiplying both sides by $\cos \frac{p\pi x}{L}$ (p being an integer), integrating both sides with respect to x from 0 to L, and taking advantage of the fact that cosine functions are orthogonal with one another.

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

A_0 is obtained just by integrating both sides of the series expansion with respect to x from 0 to L.

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

Part (a)

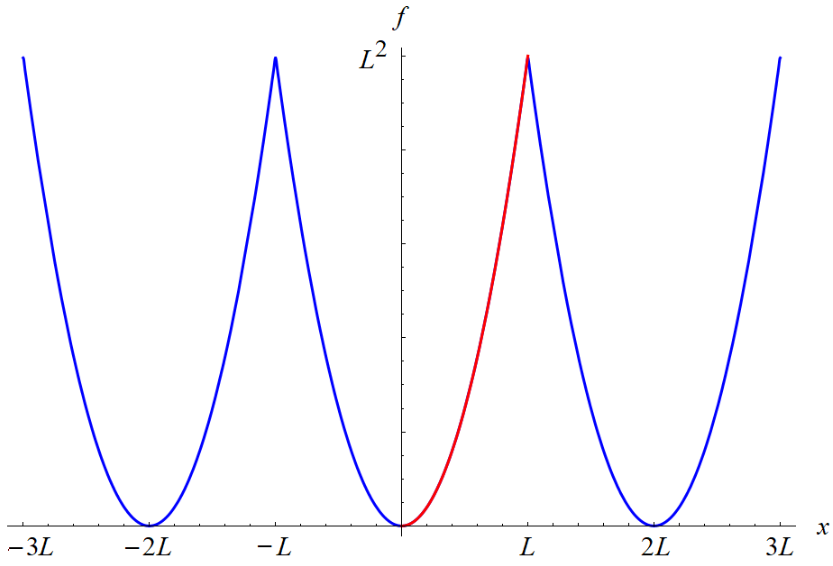
For $f(x) = x^2$, the coefficients are

$$A_0 = \frac{1}{L} \int_0^L x^2\, dx = \frac{L^2}{3}$$

and

$$A_n = \frac{2}{L} \int_0^L x^2 \cos \frac{n\pi x}{L}\, dx = \frac{4(-1)^n L^2}{n^2 \pi^2}.$$

Below is a graph of the function and its even extension to the whole line.



Part (b)

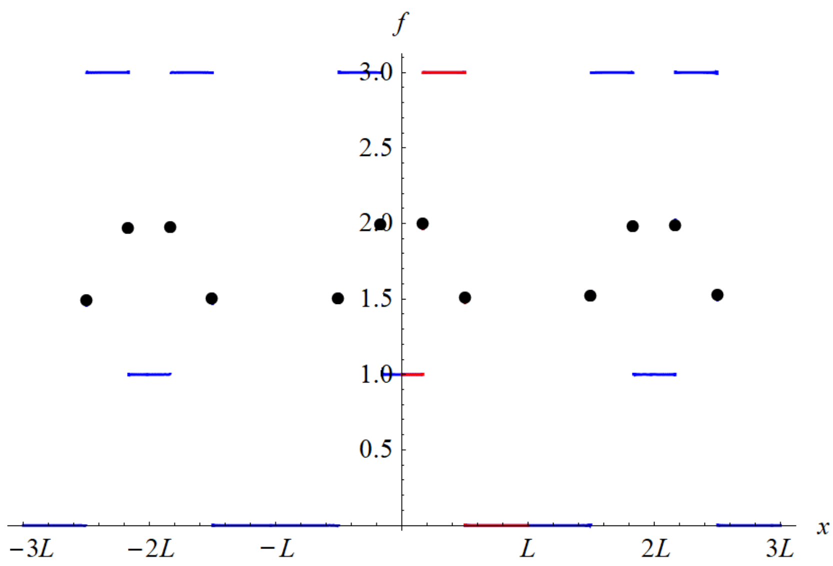
For $f(x) = 1$ if $x < L/6$ and $f(x) = 3$ if $L/6 < x < L/2$ and $f(x) = 0$ if $x > L/2$, the coefficients are

$$A_0 = \frac{1}{L}\left(\int_0^{L/6} dx + \int_{L/6}^{L/2} 3\,dx + \int_{L/2}^{L} 0\,dx\right) = \frac{7}{6}$$

and

$$A_n = \frac{2}{L}\left(\int_0^{L/6} \cos\frac{n\pi x}{L}\,dx + \int_{L/6}^{L/2} 3\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 0\cos\frac{n\pi x}{L}\,dx\right) = \frac{2}{n\pi}\left(3\sin\frac{n\pi}{2} - 2\sin\frac{n\pi}{6}\right).$$

Below is a graph of the function and its even extension to the whole line.



Part (c)

For $f(x) = 0$ if $x < L/2$ and $f(x) = x$ if $x > L/2$, the coefficients are

$$A_0 = \frac{1}{L}\left(\int_0^{L/2} 0\,dx + \int_{L/2}^{L} x\,dx\right) = \frac{3L}{8}$$

and

$$A_n = \frac{2}{L}\left(\int_0^{L/2} 0\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} x\cos\frac{n\pi x}{L}\,dx\right) = \frac{L}{n^2\pi^2}\left[2(-1)^n - 2\cos\frac{n\pi}{2} - n\pi\sin\frac{n\pi}{2}\right].$$

Below is a graph of the function and its even extension to the whole line.