Exercise 3.3.6

For the following functions, sketch the Fourier cosine series of $f(x)$. Also, roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier cosine series:

(a) $f(x) = x$ [Use formulas (3.3.22) and (3.3.23).]

(b) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ [Use carefully formulas (3.2.6) and (3.2.7).]

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ [*Hint*: Add the functions in parts (b) and (c).]

Solution

Assume that $f(x)$ is a piecewise smooth function on the interval $0 \leq x \leq L$. The even extension of $f(x)$ to the whole line with period $2L$ is given by the Fourier cosine series expansion,

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},$$

at points where $f(x)$ is continuous and by the average of the left-hand and right-hand limits at points of discontinuity. The coefficients A_n are obtained by multiplying both sides by $\cos \frac{p\pi x}{L}$ (p being an integer), integrating both sides with respect to x from 0 to L, and taking advantage of the fact that cosine functions are orthogonal with one another.

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx$$

A_0 is obtained just by integrating both sides of the series expansion with respect to x from 0 to L.

$$A_0 = \frac{1}{L} \int_0^L f(x) \, dx$$

Part (a)

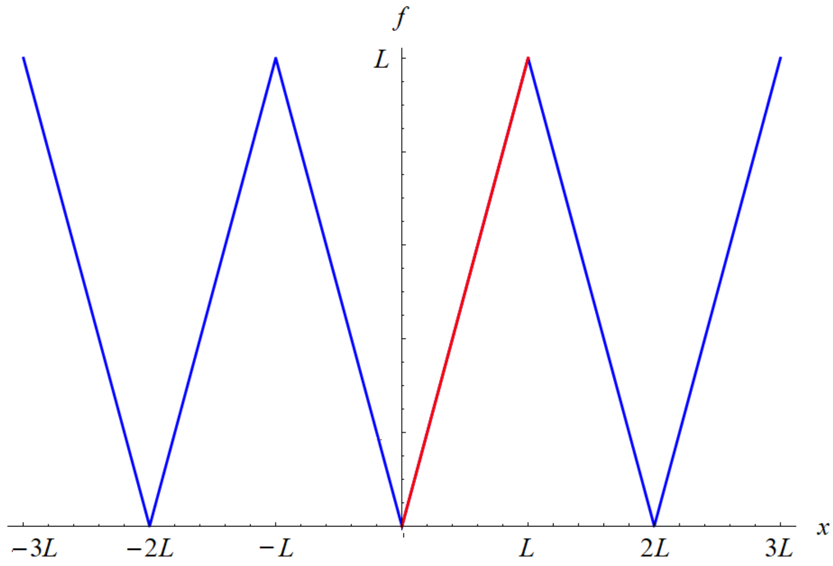
For $f(x) = x$, the coefficients are

$$A_0 = \frac{1}{L} \int_0^L x \, dx = \frac{L}{2}$$

and

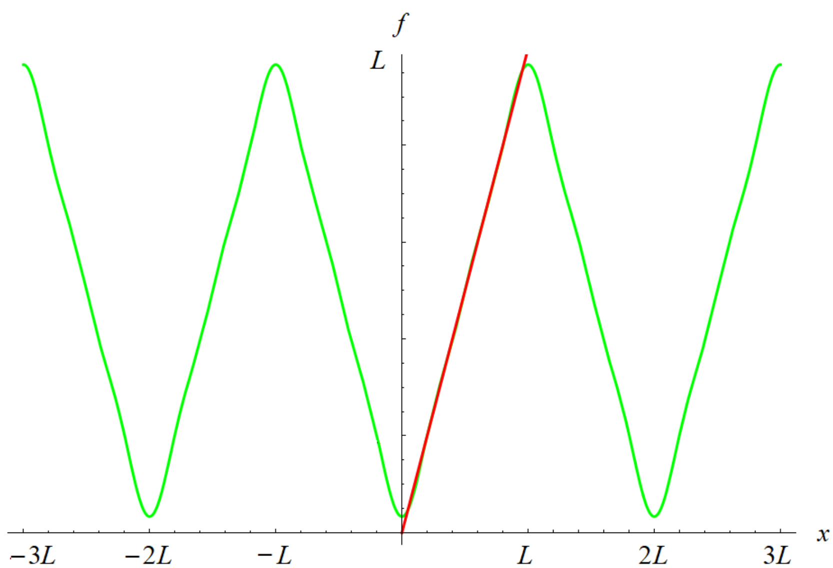
$$A_n = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} \, dx = \frac{2[-1 + (-1)^n]L}{n^2\pi^2}.$$

Below is a graph of the function and its even extension to the whole line.



Below is a graph using the first five terms in the infinite series:

$$f(x) \approx A_0 + \sum_{n=1}^{5} A_n \cos \frac{n\pi x}{L}.$$



Part (b)

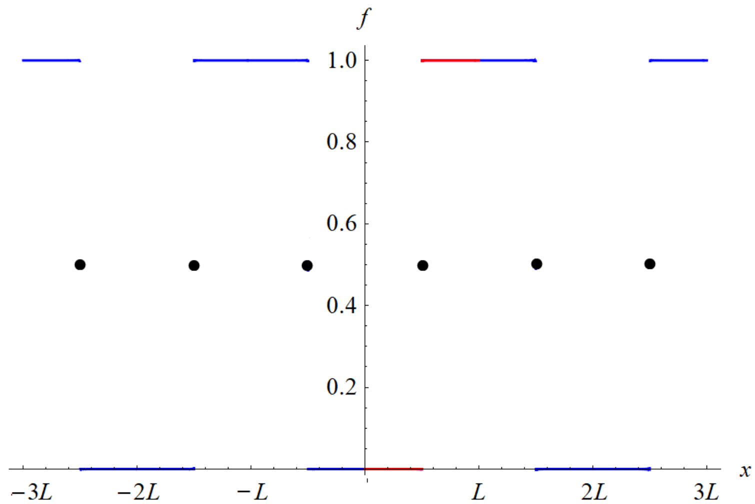
For $f(x) = 0$ if $x < L/2$ and $f(x) = 1$ if $x > L/2$, the coefficients are

$$A_0 = \frac{1}{L}\left(\int_0^{L/2} 0\,dx + \int_{L/2}^{L} dx\right) = \frac{1}{2}$$

and

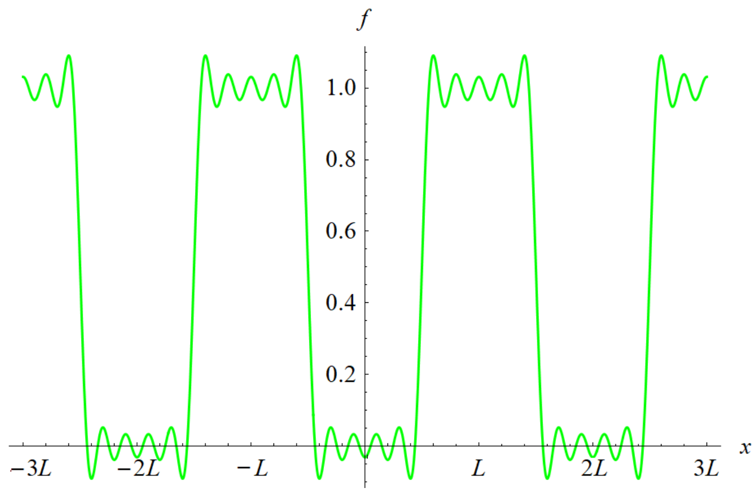
$$A_n = \frac{2}{L}\left(\int_0^{L/2} 0\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} \cos\frac{n\pi x}{L}\,dx\right) = -\frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Below is a graph of the function and its even extension to the whole line.



Below is a graph using the first ten terms in the infinite series:

$$f(x) \approx A_0 + \sum_{n=1}^{10} A_n \cos\frac{n\pi x}{L}.$$



Part (c)

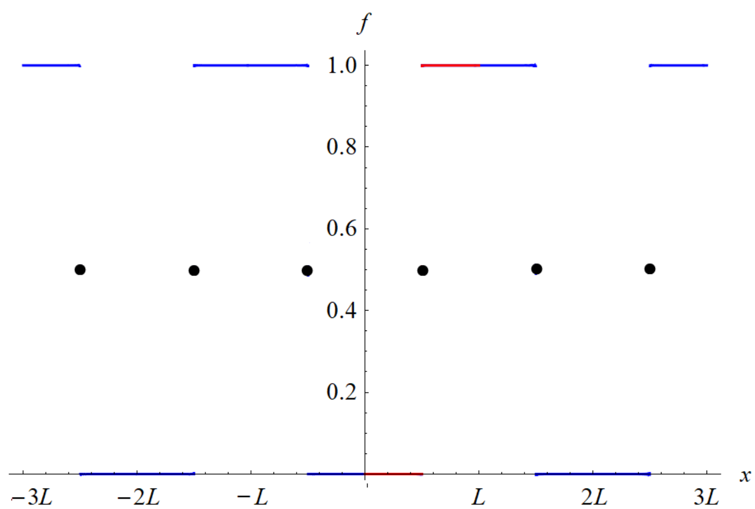
For $f(x) = 0$ if $x < L/2$ and $f(x) = 1$ if $x > L/2$, the coefficients are

$$A_0 = \frac{1}{L}\left(\int_0^{L/2} 0\,dx + \int_{L/2}^{L} dx\right) = \frac{1}{2}$$

and

$$A_n = \frac{2}{L}\left(\int_0^{L/2} 0\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} \cos\frac{n\pi x}{L}\,dx\right) = -\frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Below is a graph of the function and its even extension to the whole line.



Below is a graph using the first ten terms in the infinite series:

$$f(x) \approx A_0 + \sum_{n=1}^{10} A_n \cos\frac{n\pi x}{L}.$$