Exercise 3.3.9

What is the sum of the Fourier sine series of $f(x)$ and the Fourier cosine series of $f(x)$? [What is the sum of the even and odd extensions of $f(x)$?]

Solution

The Fourier sine series expansion of $f(x)$, a piecewise smooth function defined on $0 \leq x \leq L$, is given by

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

It represents the $2L$-periodic odd extension of $f(x)$ to the whole line ($-\infty < x < \infty$). That is,

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = \begin{cases} f(x) & 2mL < x < (2m+1)L \\ -f(-x) & (2m-1)L < x < 2mL \end{cases} \tag{1}$$

for any integer m. On the other hand, the Fourier cosine series expansion of this same function on $0 \leq x \leq L$ is given by

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L},$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x) \, dx$$
$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx.$$

It represents the $2L$-periodic even extension of $f(x)$ to the whole line ($-\infty < x < \infty$). That is,

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = \begin{cases} f(x) & 2mL < x < (2m+1)L \\ f(-x) & (2m-1)L < x < 2mL \end{cases} \tag{2}$$

for any integer m. Add the respective sides of equations (1) and (2).

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = \begin{cases} 2f(x) & 2mL < x < (2m+1)L \\ 0 & (2m-1)L < x < 2mL \end{cases}$$

This result holds where $f(x)$ is continuous; at points of discontinuity the average of the left-hand and right-hand limits is taken.