Exercise 3.4.1

The integration-by-parts formula

$$\int_a^b u\frac{dv}{dx}\,dx = uv\bigg|_a^b - \int_a^b v\frac{du}{dx}\,dx$$

is known to be valid for functions $u(x)$ and $v(x)$, which are continuous and have continuous first derivatives. However, we will assume that u, v, du/dx, and dv/dx are continuous only for $a \le x \le c$ and $c \le x \le b$; we assume that all quantities may have a jump discontinuity at $x = c$.

(a) Derive an expression for $\int_a^b u\,dv/dx\,dx$ in terms of $\int_a^b v\,du/dx\,dx$.

(b) Show that this reduces to the integration-by-parts formula if u and v are continuous across $x = c$. It is not necessary for du/dx and dv/dx to be continuous at $x = c$.

Solution

This formula will be modified to take a jump discontinuity at $x = c$ into account.

$$\begin{aligned}
\int_a^b u\frac{dv}{dx}\,dx &= \int_a^{c-} u\frac{dv}{dx}\,dx + \int_{c+}^b u\frac{dv}{dx}\,dx \\
&= \left(uv\bigg|_a^{c-} - \int_a^{c-} v\frac{du}{dx}\,dx\right) + \left(uv\bigg|_{c+}^b - \int_{c+}^b v\frac{du}{dx}\,dx\right) \\
&= u(c-)v(c-) - u(a)v(a) + u(b)v(b) - u(c+)v(c+) - \left(\int_a^{c-} v\frac{du}{dx}\,dx + \int_{c+}^b v\frac{du}{dx}\,dx\right) \\
&= [u(b)v(b) - u(a)v(a)] - [u(c+)v(c+) - u(c-)v(c-)] - \int_a^b v\frac{du}{dx}\,dx \\
&= uv\bigg|_a^b - uv\bigg|_{c-}^{c+} - \int_a^b v\frac{du}{dx}\,dx
\end{aligned}$$

If u and v are continuous at $x = c$, then $u(c-) = u(c+)$ and $v(c-) = v(c+)$. This formula then reduces to the standard one for integration by parts because

$$\begin{aligned}
uv\bigg|_{c-}^{c+} &= u(c+)v(c+) - u(c-)v(c-) \\
&= 0.
\end{aligned}$$