Exercise 3.4.10

Modify Exercise 3.4.9 if instead $\partial u/\partial x(0,t) = 0$ and $\partial u/\partial x(L,t) = 0$.

Solution

Assuming that u is continuous on $0 \leq x \leq L$, it has a Fourier cosine series expansion.

$$u(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t) \cos \frac{n\pi x}{L} \tag{1}$$

Because $\partial u/\partial t$ is piecewise smooth, the series can be differentiated with respect to t term by term.

$$\frac{\partial u}{\partial t} = A_0'(t) + \sum_{n=1}^{\infty} A_n'(t) \cos \frac{n\pi x}{L}$$

And because u is continuous, the cosine series can be differentiated with respect to x term by term.

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \left(-\frac{n\pi}{L}\right) A_n(t) \sin \frac{n\pi x}{L}$$

Since u_x is also continuous on $0 \leq x \leq L$ and $u_x(0,t) = u_x(L,t) = 0$, term-by-term differentiation of this sine series with respect to x is justified.

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{L^2}\right) A_n(t) \cos \frac{n\pi x}{L}$$

Substitute these infinite series into the PDE.

$$A_0'(t) + \sum_{n=1}^{\infty} A_n'(t) \cos \frac{n\pi x}{L} = k \sum_{n=1}^{\infty} \left(-\frac{n^2\pi^2}{L^2}\right) A_n(t) \cos \frac{n\pi x}{L} + q(x,t)$$

Bring both series to the left side.

$$A_0'(t) + \sum_{n=1}^{\infty} A_n'(t) \cos \frac{n\pi x}{L} + k \sum_{n=1}^{\infty} \left(\frac{n^2\pi^2}{L^2}\right) A_n(t) \cos \frac{n\pi x}{L} = q(x,t)$$

Combine the series and factor the summand.

$$A_0'(t) + \sum_{n=1}^{\infty} \left[A_n'(t) + \frac{kn^2\pi^2}{L^2} A_n(t)\right] \cos \frac{n\pi x}{L} = q(x,t) \tag{2}$$

This is the Fourier cosine series expansion of $q(x,t)$; because $q(x,t)$ is piecewise smooth, it's valid. To obtain $A_0(t)$, integrate both sides with respect to x from 0 to L.

$$\int_0^L \left\{A_0'(t) + \sum_{n=1}^{\infty} \left[A_n'(t) + \frac{kn^2\pi^2}{L^2} A_n(t)\right] \cos \frac{n\pi x}{L}\right\} dx = \int_0^L q(x,t)\, dx$$

Split up the integral on the left and bring the constants in front.

$$A_0'(t) \underbrace{\int_0^L dx}_{=L} + \sum_{n=1}^{\infty} \left[A_n'(t) + \frac{kn^2\pi^2}{L^2} A_n(t)\right] \underbrace{\int_0^L \cos \frac{n\pi x}{L}\, dx}_{=0} = \int_0^L q(x,t)\, dx$$

Evaluate the integrals.

$$A_0'(t)(L) = \int_0^L q(x,t)\,dx$$

Divide both sides by L to obtain the ODE for $A_0(t)$.

$$A_0'(t) = \frac{1}{L}\int_0^L q(x,t)\,dx$$

Integrate both sides with respect to t.

$$A_0(t) = \frac{1}{L}\int^t \int_0^L q(x,s)\,dx\,ds + C_1$$

The lower limit of integration is arbitrary and can be set to zero. C_1 will be adjusted to account for any choice that's made.

$$A_0(t) = \frac{1}{L}\int_0^t \int_0^L q(x,s)\,dx\,ds + C_1$$

To get $A_n(t)$, multiply both sides of equation (2) by $\cos\frac{p\pi x}{L}$, where p is an integer,

$$A_0'(t)\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L} = q(x,t)\cos\frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \left\{A_0'(t)\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty}\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right\}dx = \int_0^L q(x,t)\cos\frac{p\pi x}{L}\,dx$$

Split up the integral on the left and bring the constants in front.

$$A_0'(t)\underbrace{\int_0^L \cos\frac{p\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty}\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\int_0^L \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx = \int_0^L q(x,t)\cos\frac{p\pi x}{L}\,dx$$

Since the cosine functions are orthogonal, the second integral on the left is zero if $n \neq p$. Only if $n = p$ is it nonzero.

$$\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\int_0^L \cos^2\frac{n\pi x}{L}\,dx = \int_0^L q(x,t)\cos\frac{n\pi x}{L}\,dx$$

Evaluate the integral on the left.

$$\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\frac{L}{2} = \int_0^L q(x,t)\cos\frac{n\pi x}{L}\,dx$$

The ODE that $A_n(t)$ satisfies is then

$$A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t) = \frac{2}{L}\int_0^L q(x,t)\cos\frac{n\pi x}{L}\,dx,$$

which is a first-order linear inhomogeneous ODE, so it can be solved by using an integrating factor I.

$$I = \exp\left(\int^t \frac{kn^2\pi^2}{L^2}\,ds\right) = \exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

Multiply both sides of the ODE by I.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)A_n'(t) + \frac{kn^2\pi^2}{L^2}\exp\left(\frac{kn^2\pi^2}{L^2}t\right)A_n(t) = \left[\frac{2}{L}\int_0^L q(x,t)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

The left side can be written as $d/dt(IA_n)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(\frac{kn^2\pi^2}{L^2}t\right)A_n(t)\right] = \left[\frac{2}{L}\int_0^L q(x,t)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

Integrate both sides with respect to t.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)A_n(t) = \int^t \left[\frac{2}{L}\int_0^L q(x,s)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + C_2$$

The lower limit of integration is arbitrary and can be set to zero. C_2 will be adjusted to account for any choice that's made.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)A_n(t) = \int_0^t \left[\frac{2}{L}\int_0^L q(x,s)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + C_2$$

Solve for $A_n(t)$.

$$A_n(t) = \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\left\{\int_0^t \left[\frac{2}{L}\int_0^L q(x,s)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + C_2\right\}$$

An initial condition is needed to determine C_1 and C_2. Use equation (1) along with $u(x,0) = f(x)$ to determine them.

$$u(x,0) = A_0(0) + \sum_{n=1}^{\infty} A_n(0)\cos\frac{n\pi x}{L} = f(x)$$

These coefficients are known,

$$A_0(0) = \frac{1}{L}\int_0^L f(x)\,dx$$

$$A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,$$

so C_1 and C_2 are as well.

$$A_0(0) = C_1 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$A_n(0) = C_2 = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Therefore,

$$\begin{aligned}
A_0(t) &= \frac{1}{L}\int_0^t \int_0^L q(x,s)\,dx\,ds + \frac{1}{L}\int_0^L f(x)\,dx \\
&= \frac{1}{L}\left[\int_0^t \int_0^L q(x,s)\,dx\,ds + \int_0^L f(x)\,dx\right] \\
A_n(t) &= \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\left\{\int_0^t \left[\frac{2}{L}\int_0^L q(x,s)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx\right\} \\
&= \frac{2}{L}\left\{\int_0^t \int_0^L q(x,s)\cos\frac{n\pi x}{L}\exp\left(\frac{kn^2\pi^2}{L^2}s\right)dx\,ds + \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx\right\}\exp\left(-\frac{kn^2\pi^2}{L^2}t\right),
\end{aligned}$$

and the solution to the PDE is

$$\begin{aligned}
u(x,t) &= A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos\frac{n\pi x}{L} \\
&= \frac{1}{L}\left[\int_0^t \int_0^L q(x,s)\,dx\,ds + \int_0^L f(x)\,dx\right] \\
&\quad + \sum_{n=1}^{\infty}\frac{2}{L}\left\{\int_0^t \int_0^L q(x,s)\cos\frac{n\pi x}{L}\exp\left(\frac{kn^2\pi^2}{L^2}s\right)dx\,ds + \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx\right\}\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\cos\frac{n\pi x}{L}.
\end{aligned}$$