Exercise 3.4.4

Suppose that $f(x)$ and df/dx are piecewise smooth.

(a) Prove that the Fourier sine series of a continuous function $f(x)$ can be differentiated term by term only if $f(0) = 0$ and $f(L) = 0$.

(b) Prove that the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

Solution

Part (a)

If $f(x)$ is piecewise smooth on $0 \leq x \leq L$, then it has a Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

The derivative of $f(x)$ is expected to be a series of cosines; because df/dx is also piecewise smooth, it has a Fourier cosine series.

$$\frac{df}{dx} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \tag{1}$$

The aim is to show that

$$A_0 = 0 \quad \text{and} \quad A_n = \frac{n\pi}{L} B_n$$

and to determine the conditions for which these formulas hold. To get A_0, integrate both sides of equation (1) with respect to x from 0 to L.

$$\begin{aligned}
\int_0^L \frac{df}{dx}\,dx &= \int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}\right) dx \\
&= A_0 \int_0^L dx + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos \frac{n\pi x}{L}\,dx}_{=\,0} \\
&= A_0(L)
\end{aligned}$$

Solve for A_0.

$$\begin{aligned}
A_0 &= \frac{1}{L} \int_0^L \frac{df}{dx}\,dx \\
&= \frac{1}{L}[f(L) - f(0)]
\end{aligned}$$

Only if $f(L) = f(0)$ does $A_0 = 0$.

To get A_n, multiply both sides of equation (1) by $\cos\frac{p\pi x}{L}$, where p is an integer,

$$\frac{df}{dx}\cos\frac{p\pi x}{L} = A_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\begin{aligned}\int_0^L \frac{df}{dx}\cos\frac{p\pi x}{L}\,dx &= \int_0^L\left(A_0\cos\frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\right)dx \\ &= A_0\underbrace{\int_0^L \cos\frac{p\pi x}{L}\,dx}_{=\,0} + \sum_{n=1}^{\infty} A_n\int_0^L \cos\frac{n\pi x}{L}\cos\frac{p\pi x}{L}\,dx\end{aligned}$$

Because the cosine functions are orthogonal, this second integral on the right is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$\begin{aligned}\int_0^L \frac{df}{dx}\cos\frac{n\pi x}{L}\,dx &= A_n\int_0^L \cos^2\frac{n\pi x}{L}\,dx \\ &= A_n\left(\frac{L}{2}\right)\end{aligned}$$

Solve for A_n.

$$\begin{aligned}A_n &= \frac{2}{L}\int_0^L \frac{df}{dx}\cos\frac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\left[f(x)\cos\frac{n\pi x}{L}\bigg|_0^L - \int_0^L f(x)\frac{d}{dx}\left(\cos\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{2}{L}\left[f(L)\cos n\pi - f(0) - \int_0^L f(x)\left(-\frac{n\pi}{L}\sin\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{2}{L}[f(L)(-1)^n - f(0)] + \frac{n\pi}{L}\left[\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right] \\ &= \frac{2}{L}[f(L)(-1)^n - f(0)] + \frac{n\pi}{L}B_n\end{aligned}$$

Only if $f(L) = f(0) = 0$ does $A_n = (n\pi/L)B_n$. Therefore, the Fourier sine series can be differentiated term by term if f is continuous and only if $f(L) = f(0) = 0$.

Part (b)

If $f(x)$ is piecewise smooth on $0 \leq x \leq L$, then it has a Fourier cosine series.

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

The derivative of $f(x)$ is expected to be a series of sines; because df/dx is also piecewise smooth, it has a Fourier sine series.

$$\frac{df}{dx} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \tag{2}$$

The aim is to show that

$$B_n = -\frac{n\pi}{L} A_n$$

and to determine the conditions for which this formula holds. Multiply both sides of equation (2) by $\sin \frac{p\pi x}{L}$, where p is an integer,

$$\frac{df}{dx} \sin \frac{p\pi x}{L} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\begin{aligned}\int_0^L \frac{df}{dx} \sin \frac{p\pi x}{L}\, dx &= \int_0^L \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\, dx \\ &= \sum_{n=1}^{\infty} B_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L}\, dx\end{aligned}$$

Because the sine functions are orthogonal with one another, this integral on the right is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$\begin{aligned}\int_0^L \frac{df}{dx} \sin \frac{n\pi x}{L}\, dx &= B_n \int_0^L \sin^2 \frac{n\pi x}{L}\, dx \\ &= B_n \left(\frac{L}{2}\right)\end{aligned}$$

Solve for B_n.

$$\begin{aligned}B_n &= \frac{2}{L} \int_0^L \frac{df}{dx} \sin \frac{n\pi x}{L}\, dx \\ &= \frac{2}{L}\left[f(x) \sin \frac{n\pi x}{L}\bigg|_0^L - \int_0^L f(x) \frac{d}{dx}\left(\sin \frac{n\pi x}{L}\right) dx\right] \\ &= \frac{2}{L}\left[f(L) \sin n\pi - \int_0^L f(x) \left(\frac{n\pi}{L} \cos \frac{n\pi x}{L}\right) dx\right] \\ &= \frac{2}{L}\left[-\frac{n\pi}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx\right] \\ &= -\frac{n\pi}{L}\left[\frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx\right] \\ &= -\frac{n\pi}{L} A_n\end{aligned}$$

Therefore, the Fourier cosine series can be differentiated term by term if f is continuous.