Exercise 3.4.5

Using (3.3.13) determine the Fourier cosine series of \( \sin \frac{\pi x}{L} \).

Solution

Because \( \sin \frac{\pi x}{L} \) is a continuous function (assumed to be defined on \( 0 \leq x \leq L \)), it has a Fourier cosine series.

\[
\sin \frac{\pi x}{L} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}
\]  

This series on the right is the \( 2L \)-periodic even extension of \( \sin \frac{\pi x}{L} \) to the whole line. To determine \( A_0 \), integrate both sides with respect to \( x \) from 0 to \( L \).

\[
\int_0^L \sin \frac{\pi x}{L} \, dx = \int_0^L \left( A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \right) \, dx
\]

Split up the integral on the right and bring the constants in front.

\[
\int_0^L \sin \frac{\pi x}{L} \, dx = A_0 \int_0^L \cos \frac{p\pi x}{L} \, dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L} \, dx
\]

Evaluate the integrals.

\[
\frac{2L}{\pi} = A_0(L)
\]

Solve for \( A_0 \).

\[
A_0 = \frac{2}{\pi}
\]

To determine \( A_n \), multiply both sides of equation (1) by \( \cos \frac{p\pi x}{L} \), where \( p \) is an integer,

\[
\sin \frac{\pi x}{L} \cos \frac{p\pi x}{L} = A_0 \cos \frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}
\]

and then integrate both sides with respect to \( x \) from 0 to \( L \).

\[
\int_0^L \sin \frac{\pi x}{L} \cos \frac{p\pi x}{L} \, dx = \int_0^L \left( A_0 \cos \frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L} \right) \, dx
\]

Split up the integral on the right and bring the constants in front.

\[
\int_0^L \sin \frac{\pi x}{L} \cos \frac{p\pi x}{L} \, dx = A_0 \int_0^L \cos \frac{p\pi x}{L} \, dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L} \, dx \]

Because the cosine functions are orthogonal, this second integral on the right is zero if \( n \neq p \). Only if \( n = p \) does it yield a nonzero value.

\[
\int_0^L \sin \frac{\pi x}{L} \cos \frac{n\pi x}{L} \, dx = A_n \int_0^L \cos^2 \frac{n\pi x}{L} \, dx = A_n \left( \frac{L}{2} \right)
\]
Solve for $A_n$. 

\[
A_n = \frac{2}{L} \int_0^L \sin \frac{\pi x}{L} \cos \frac{n\pi x}{L} \, dx
\]

\[
= \frac{2}{L} \int_0^L \frac{1}{2} \left[ \sin \left( \frac{\pi x}{L} + \frac{n\pi x}{L} \right) + \sin \left( \frac{\pi x}{L} - \frac{n\pi x}{L} \right) \right] \, dx
\]

\[
= \frac{1}{L} \left[ \int_0^L \sin \frac{(1+n)\pi x}{L} \, dx + \int_0^L \sin \frac{(1-n)\pi x}{L} \, dx \right] = 0 \text{ if } n = 1
\]

\[
= \frac{1}{L} \left[ \frac{L[1 + (-1)^n]}{(1+n)\pi} + \frac{L[1 + (-1)^n]}{(1-n)\pi} \right] \text{ if } n \neq 1
\]

\[
= \frac{2[1 + (-1)^n]}{(1 - n^2)\pi}
\]

\[
= -\frac{2[1 + (-1)^n]}{(n^2 - 1)\pi}
\]

\[
= \begin{cases} 
0 & n = 1 \\
-\frac{2[1 + (-1)^n]}{(n^2 - 1)\pi} & n \neq 1
\end{cases}
\]

As a result, the Fourier cosine series expansion of $\sin \frac{\pi x}{L}$ is

\[
\sin \frac{\pi x}{L} = \frac{2}{\pi} + \sum_{n=2}^{\infty} \left\{ -\frac{2[1 + (-1)^n]}{(n^2 - 1)\pi} \right\} \cos \frac{n\pi x}{L}
\]

\[
= \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{1 + (-1)^n}{n^2 - 1} \cos \frac{n\pi x}{L}.
\]

Notice that the summand vanishes if $n$ is odd. The infinite series can then be simplified (that is, made to converge faster) by summing over the even integers only. Substitute $n = 2k$ in the series.

\[
\sin \frac{\pi x}{L} = \frac{2}{\pi} - \frac{2}{\pi} \sum_{k=2}^{\infty} \frac{1 + (-1)^k}{(2k)^2 - 1} \cos \frac{2k\pi x}{L}
\]

\[
= \frac{2}{\pi} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{2}{4k^2 - 1} \cos \frac{2k\pi x}{L}
\]

\[
= \frac{2}{\pi} \left( 1 - \sum_{k=1}^{\infty} \frac{2}{4k^2 - 1} \cos \frac{2k\pi x}{L} \right)
\]

Therefore,

\[
\sin \frac{\pi x}{L} = \frac{2}{\pi} \left( 1 - 2 \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \cos \frac{2k\pi x}{L} \right).
\]