Exercise 3.4.5

Using (3.3.13) determine the Fourier cosine series of $\sin \pi x/L$.

Solution

Because $\sin \pi x/L$ is a continuous function (assumed to be defined on $0 \le x \le L$), it has a Fourier cosine series.

$$\sin \frac{\pi x}{L} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \tag{1}$$

This series on the right is the $2L$-periodic even extension of $\sin \pi x/L$ to the whole line. To determine A_0, integrate both sides with respect to x from 0 to L.

$$\int_0^L \sin \frac{\pi x}{L}\, dx = \int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \right) dx$$

Split up the integral on the right and bring the constants in front.

$$\int_0^L \sin \frac{\pi x}{L}\, dx = A_0 \int_0^L dx + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos \frac{n\pi x}{L}\, dx}_{=\,0}$$

Evaluate the integrals.

$$\frac{2L}{\pi} = A_0(L)$$

Solve for A_0.

$$A_0 = \frac{2}{\pi}$$

To determine A_n, multiply both sides of equation (1) by $\cos \frac{p\pi x}{L}$, where p is an integer,

$$\sin \frac{\pi x}{L} \cos \frac{p\pi x}{L} = A_0 \cos \frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \sin \frac{\pi x}{L} \cos \frac{p\pi x}{L}\, dx = \int_0^L \left(A_0 \cos \frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L} \right) dx$$

Split up the integral on the right and bring the constants in front.

$$\int_0^L \sin \frac{\pi x}{L} \cos \frac{p\pi x}{L}\, dx = A_0 \underbrace{\int_0^L \cos \frac{p\pi x}{L}\, dx}_{=\,0} + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}\, dx$$

Because the cosine functions are orthogonal, this second integral on the right is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero value.

$$\begin{aligned} \int_0^L \sin \frac{\pi x}{L} \cos \frac{n\pi x}{L}\, dx &= A_n \int_0^L \cos^2 \frac{n\pi x}{L}\, dx \\ &= A_n \left(\frac{L}{2} \right) \end{aligned}$$

Solve for A_n.

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx \\
&= \frac{2}{L}\int_0^L \frac{1}{2}\left[\sin\left(\frac{\pi x}{L}+\frac{n\pi x}{L}\right)+\sin\left(\frac{\pi x}{L}-\frac{n\pi x}{L}\right)\right]dx \\
&= \frac{1}{L}\left[\int_0^L \sin\frac{(1+n)\pi x}{L}\,dx + \int_0^L \sin\frac{(1-n)\pi x}{L}\,dx\right] = 0 \quad \text{if } n=1 \\
&= \frac{1}{L}\left[\frac{L[1+(-1)^n]}{(1+n)\pi}+\frac{L[1+(-1)^n]}{(1-n)\pi}\right] \quad \text{if } n\neq 1 \\
&= \frac{2[1+(-1)^n]}{(1-n^2)\pi} \\
&= -\frac{2[1+(-1)^n]}{(n^2-1)\pi} \\
&= \begin{cases} 0 & n=1 \\ -\dfrac{2[1+(-1)^n]}{(n^2-1)\pi} & n\neq 1 \end{cases}
\end{aligned}$$

As a result, the Fourier cosine series expansion of $\sin \pi x/L$ is

$$\begin{aligned}
\sin\frac{\pi x}{L} &= \frac{2}{\pi} + \sum_{n=2}^{\infty}\left\{-\frac{2[1+(-1)^n]}{(n^2-1)\pi}\right\}\cos\frac{n\pi x}{L} \\
&= \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}\cos\frac{n\pi x}{L}.
\end{aligned}$$

Notice that the summand vanishes if n is odd. The infinite series can then be simplified (that is, made to converge faster) by summing over the even integers only. Substitute $n=2k$ in the series.

$$\begin{aligned}
\sin\frac{\pi x}{L} &= \frac{2}{\pi} - \frac{2}{\pi}\sum_{2k=2}^{\infty}\frac{1+(-1)^{2k}}{(2k)^2-1}\cos\frac{2k\pi x}{L} \\
&= \frac{2}{\pi} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{2}{4k^2-1}\cos\frac{2k\pi x}{L} \\
&= \frac{2}{\pi}\left(1-\sum_{k=1}^{\infty}\frac{2}{4k^2-1}\cos\frac{2k\pi x}{L}\right)
\end{aligned}$$

Therefore,

$$\sin\frac{\pi x}{L} = \frac{2}{\pi}\left(1-2\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos\frac{2k\pi x}{L}\right).$$