Exercise 3.4.6

There are some things wrong in the following demonstration. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of \( e^x \):

\[
e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}.
\]  \hfill (3.4.22)

Differentiating yields

\[
e^x = - \sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L},
\]

the Fourier sine series of \( e^x \). Differentiating again yields

\[
e^x = - \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right)^2 A_n \cos \frac{n\pi x}{L}.
\]  \hfill (3.4.23)

Since Equations (3.4.22) and (3.4.23) give the Fourier cosine series of \( e^x \), they must be identical. Thus,

\[\begin{align*}
A_0 &= 0 \\
A_n &= 0
\end{align*}\]

(obviously wrong!).

By correcting the mistakes, you should be able to obtain \( A_0 \) and \( A_n \) without using the typical technique, that is, \( A_n = 2/L \int_0^L e^x \cos n\pi x/L \, dx \).

Solution

\( e^x \) is a continuous function on \( 0 \leq x \leq L \), so it has a Fourier cosine series expansion.

\[
e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \] \hfill (1)

Because \( e^x \) is continuous, there’s no problem differentiating its cosine series with respect to \( x \) term by term.

\[
e^x = \sum_{n=1}^{\infty} \left( - \frac{n\pi}{L} A_n \right) \sin \frac{n\pi x}{L}
\]

This is now a sine series, so differentiating term by term is not justified because \( e^0 \neq 0 \) and \( e^L \neq 0 \). Rather, use Eq. 3.4.13 on page 117.

\[
e^x = \frac{1}{L} (e^L - 1) + \sum_{n=1}^{\infty} \left[ \frac{n\pi}{L} \left( - \frac{n\pi}{L} A_n \right) + \frac{2}{L} \left[ (-1)^n e^L - 1 \right] \right] \cos \frac{n\pi x}{L}
\] \hfill (2)

Comparing equations (1) and (2) gives

\[
A_0 = \frac{1}{L} (e^L - 1)
\]

\[
A_n = \frac{n\pi}{L} \left( - \frac{n\pi}{L} A_n \right) + \frac{2}{L} \left[ (-1)^n e^L - 1 \right] \rightarrow A_n = \frac{2L (-1)^n e^L - 1}{n^2\pi^2 + L^2}.
\]

www.stemjock.com