Exercise 3.4.8

Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to

$$\partial u/\partial x(0,t) = 0, \quad \partial u/\partial x(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x).$$

Solve in the following way. Look for the solution as a Fourier cosine series. Assume that u and $\partial u/\partial x$ are continuous and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Justify all differentiations of infinite series.

Solution

Assuming that u is continuous on $0 \le x \le L$, it has a Fourier cosine series expansion.

$$u(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos\frac{n\pi x}{L} \tag{1}$$

Because $\partial u/\partial t$ is piecewise smooth, the series can be differentiated with respect to t term by term.

$$\frac{\partial u}{\partial t} = A_0'(t) + \sum_{n=1}^{\infty} A_n'(t)\cos\frac{n\pi x}{L}$$

And because u is continuous, the cosine series can be differentiated with respect to x term by term.

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} A_n(t)\left(-\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L}$$

Since $u_x(0,t) = u_x(L,t) = 0$, term-by-term differentiation of this sine series with respect to x is justified.

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} A_n(t)\left(-\frac{n^2\pi^2}{L^2}\right)\cos\frac{n\pi x}{L}$$

Substitute these infinite series into the PDE.

$$A_0'(t) + \sum_{n=1}^{\infty} A_n'(t)\cos\frac{n\pi x}{L} = k\sum_{n=1}^{\infty} A_n(t)\left(-\frac{n^2\pi^2}{L^2}\right)\cos\frac{n\pi x}{L}$$

Bring all terms to the left side.

$$A_0'(t) + \sum_{n=1}^{\infty} A_n'(t)\cos\frac{n\pi x}{L} + k\sum_{n=1}^{\infty} A_n(t)\left(\frac{n^2\pi^2}{L^2}\right)\cos\frac{n\pi x}{L} = 0$$

Combine the series.

$$A_0'(t) + \sum_{n=1}^{\infty}\left[A_n'(t)\cos\frac{n\pi x}{L} + A_n(t)\left(\frac{kn^2\pi^2}{L^2}\right)\cos\frac{n\pi x}{L}\right] = 0$$

Factor the summand.

$$A_0'(t) + \sum_{n=1}^{\infty}\left[A_n'(t) + \frac{kn^2\pi^2}{L^2}A_n(t)\right]\cos\frac{n\pi x}{L} = 0$$

Since the right side is zero, the coefficients must all be zero.

$$A_0'(t) = 0$$
$$A_n'(t) + \frac{kn^2\pi^2}{L^2} A_n(t) = 0$$

Solve each ODE for $A_0(t)$ and $A_n(t)$.

$$A_0(t) = C_1$$
$$A_n(t) = C_2 \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)$$

In order to determine these constants of integration, initial conditions are needed. Use equation (1) along with $u(x,0) = f(x)$ to obtain them.

$$u(x,0) = A_0(0) + \sum_{n=1}^{\infty} A_n(0) \cos\frac{n\pi x}{L} = f(x)$$

This is the Fourier cosine series expansion of $f(x)$. As long as f is continuous, or at the very least piecewise smooth, then it's valid. The coefficients are known,

$$A_0(0) = \frac{1}{L}\int_0^L f(x)\,dx = C_1$$
$$A_n(0) = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = C_2,$$

so C_1 and C_2 are as well. Therefore,

$$\begin{aligned} u(x,t) &= A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos\frac{n\pi x}{L} \\ &= \left[\frac{1}{L}\int_0^L f(x)\,dx\right] + \sum_{n=1}^{\infty}\left\{\left[\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx\right]\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\right\}\cos\frac{n\pi x}{L}. \end{aligned}$$