Exercise 3.4.9

Consider the heat equation with a known source \( q(x, t) \):

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + q(x, t) \quad \text{with} \quad u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.
\]

Assume that \( q(x, t) \) (for each \( t > 0 \)) is a piecewise smooth function of \( x \). Also assume that \( u \) and \( \frac{\partial u}{\partial x} \) are continuous functions of \( x \) (for \( t > 0 \)) and \( \frac{\partial^2 u}{\partial x^2} \) and \( \frac{\partial u}{\partial t} \) are piecewise smooth. Thus,

\[
u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{L}.
\]

Justify spatial term-by-term differentiation. What ordinary differential equation does \( b_n(t) \) satisfy? Do not solve this differential equation.

Solution

Assuming that \( u \) is continuous on \( 0 \leq x \leq L \), it has a Fourier sine series expansion.

\[
u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{L} \quad (1)
\]

Because \( \frac{\partial u}{\partial t} \) is piecewise smooth, the series can be differentiated with respect to \( t \) term by term.

\[
\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B'_n(t) \sin \frac{n\pi x}{L}
\]

And because \( u \) is continuous and \( u(0, t) = u(L, t) = 0 \), the sine series can be differentiated with respect to \( x \) term by term.

\[
\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t) \cos \frac{n\pi x}{L}
\]

Since \( u_x \) is also continuous on \( 0 \leq x \leq L \), term-by-term differentiation of this cosine series with respect to \( x \) is justified.

\[
\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \left( -\frac{n^2 \pi^2}{L^2} \right) B_n(t) \sin \frac{n\pi x}{L}
\]

Substitute these infinite series into the PDE.

\[
\sum_{n=1}^{\infty} B'_n(t) \sin \frac{n\pi x}{L} = k \sum_{n=1}^{\infty} \left( -\frac{n^2 \pi^2}{L^2} \right) B_n(t) \sin \frac{n\pi x}{L} + q(x, t)
\]

Bring both series to the left side.

\[
\sum_{n=1}^{\infty} B'_n(t) \sin \frac{n\pi x}{L} + k \sum_{n=1}^{\infty} \left( \frac{n^2 \pi^2}{L^2} \right) B_n(t) \sin \frac{n\pi x}{L} = q(x, t)
\]

Combine the series and factor the summand.

\[
\sum_{n=1}^{\infty} \left[ B'_n(t) + k \frac{n^2 \pi^2}{L^2} B_n(t) \right] \sin \frac{n\pi x}{L} = q(x, t)
\]
This is the Fourier sine series expansion of \(q(x, t)\); because \(q(x, t)\) is piecewise smooth, it’s valid. To obtain the ODE for \(B_n(t)\), multiply both sides by \(\sin \frac{p\pi x}{L}\), where \(p\) is an integer,

\[
\sum_{n=1}^{\infty} \left[ B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t) \right] \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L} = q(x, t) \sin \frac{p\pi x}{L}
\]

and then integrate both sides with respect to \(x\) from 0 to \(L\).

\[
\int_0^L \sum_{n=1}^{\infty} \left[ B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t) \right] \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L} \, dx = \int_0^L q(x, t) \sin \frac{p\pi x}{L} \, dx
\]

Split up the integral on the left and bring the constants in front.

\[
\sum_{n=1}^{\infty} \left[ B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t) \right] \int_0^L \sin \frac{n\pi x}{L} \sin \frac{p\pi x}{L} \, dx = \int_0^L q(x, t) \sin \frac{p\pi x}{L} \, dx
\]

Since the sine functions are orthogonal, the integral on the left is zero if \(n \neq p\). Only if \(n = p\) does it yield a nonzero result.

\[
\left[ B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t) \right] \int_0^L \sin^2 \frac{n\pi x}{L} \, dx = \int_0^L q(x, t) \sin \frac{n\pi x}{L} \, dx
\]

Evaluate the integral on the left.

\[
\left[ B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t) \right] \frac{L}{2} = \int_0^L q(x, t) \sin \frac{n\pi x}{L} \, dx
\]

The ODE that \(B_n(t)\) satisfies is then

\[
B_n'(t) + \frac{kn^2\pi^2}{L^2} B_n(t) = \frac{2}{L} \int_0^L q(x, t) \sin \frac{n\pi x}{L} \, dx,
\]

which is a first-order linear inhomogeneous ODE, so it can be solved by using an integrating factor \(I\).

\[
I = \exp \left( \int^t \frac{kn^2\pi^2}{L^2} \, ds \right) = \exp \left( \frac{kn^2\pi^2}{L^2} t \right)
\]

Multiply both sides of the ODE by \(I\).

\[
\exp \left( \frac{kn^2\pi^2}{L^2} t \right) B_n'(t) + \frac{kn^2\pi^2}{L^2} \exp \left( \frac{kn^2\pi^2}{L^2} t \right) B_n(t) = \left[ \frac{2}{L} \int_0^L q(x, t) \sin \frac{n\pi x}{L} \, dx \right] \exp \left( \frac{kn^2\pi^2}{L^2} t \right)
\]

The left side can be written as \(d/dt(IB_n)\) by the product rule.

\[
\frac{d}{dt} \left[ \exp \left( \frac{kn^2\pi^2}{L^2} t \right) B_n(t) \right] = \left[ \frac{2}{L} \int_0^L q(x, t) \sin \frac{n\pi x}{L} \, dx \right] \exp \left( \frac{kn^2\pi^2}{L^2} t \right)
\]

Integrate both sides with respect to \(t\).

\[
\exp \left( \frac{kn^2\pi^2}{L^2} t \right) B_n(t) = \int^t \left[ \frac{2}{L} \int_0^L q(x, s) \sin \frac{n\pi x}{L} \, dx \right] \exp \left( \frac{kn^2\pi^2}{L^2} s \right) \, ds + C_1
\]

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The lower limit of integration is arbitrary and can be set to zero. $C_1$ will be adjusted to account for any choice that’s made.

\[ \exp \left( \frac{kn^2 \pi^2}{L^2} t \right) B_n(t) = \int_0^t \left[ \frac{2}{L} \int_0^L q(x, s) \sin \frac{n \pi x}{L} \, dx \right] \exp \left( \frac{kn^2 \pi^2}{L^2} s \right) \, ds + C_1 \]

Solve for $B_n(t)$.

\[ B_n(t) = \exp \left( - \frac{kn^2 \pi^2}{L^2} t \right) \left\{ \int_0^t \left[ \frac{2}{L} \int_0^L q(x, s) \sin \frac{n \pi x}{L} \, dx \right] \exp \left( \frac{kn^2 \pi^2}{L^2} s \right) \, ds + C_1 \right\} \]

An initial condition is needed to determine $C_1$. Use equation (1) along with $u(x, 0) = f(x)$ to determine it.

\[ u(x, 0) = \sum_{n=1}^{\infty} B_n(0) \sin \frac{n \pi x}{L} = f(x) \]

The coefficients are known,

\[ B_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx, \]

so $C_1$ is as well.

\[ B_n(0) = C_1 = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx \]

Therefore,

\[ B_n(t) = \exp \left( - \frac{kn^2 \pi^2}{L^2} t \right) \left\{ \int_0^t \left[ \frac{2}{L} \int_0^L q(x, s) \sin \frac{n \pi x}{L} \, dx \right] \exp \left( \frac{kn^2 \pi^2}{L^2} s \right) \, ds + \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx \right\} \]

and the solution to the PDE is

\[ u(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin \frac{n \pi x}{L} \]

\[ = \sum_{n=1}^{\infty} \frac{2}{L} \left\{ \int_0^t \int_0^L q(x, s) \sin \frac{n \pi x}{L} \exp \left( \frac{kn^2 \pi^2}{L^2} s \right) \, dx \, ds + \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx \right\} \exp \left( - \frac{kn^2 \pi^2}{L^2} t \right) \sin \frac{n \pi x}{L}. \]