Exercise 3.4.9

Consider the heat equation with a known source $q(x,t)$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t) \quad \text{with} \quad u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Assume that $q(x,t)$ (for each $t > 0$) is a piecewise smooth function of x. Also assume that u and $\partial u/\partial x$ are continuous functions of x (for $t > 0$) and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Thus,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L}.$$

Justify spatial term-by-term differentiation. What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this differential equation.

Solution

Assuming that u is continuous on $0 \le x \le L$, it has a Fourier sine series expansion.

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi x}{L} \tag{1}$$

Because $\partial u/\partial t$ is piecewise smooth, the series can be differentiated with respect to t term by term.

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} B_n'(t)\sin\frac{n\pi x}{L}$$

And because u is continuous and $u(0,t) = u(L,t) = 0$, the sine series can be differentiated with respect to x term by term.

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t)\cos\frac{n\pi x}{L}$$

Since u_x is also continuous on $0 \le x \le L$, term-by-term differentiation of this cosine series with respect to x is justified.

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{L^2}\right) B_n(t)\sin\frac{n\pi x}{L}$$

Substitute these infinite series into the PDE.

$$\sum_{n=1}^{\infty} B_n'(t)\sin\frac{n\pi x}{L} = k\sum_{n=1}^{\infty}\left(-\frac{n^2\pi^2}{L^2}\right) B_n(t)\sin\frac{n\pi x}{L} + q(x,t)$$

Bring both series to the left side.

$$\sum_{n=1}^{\infty} B_n'(t)\sin\frac{n\pi x}{L} + k\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2}{L^2}\right) B_n(t)\sin\frac{n\pi x}{L} = q(x,t)$$

Combine the series and factor the summand.

$$\sum_{n=1}^{\infty}\left[B_n'(t) + \frac{kn^2\pi^2}{L^2}B_n(t)\right]\sin\frac{n\pi x}{L} = q(x,t)$$

This is the Fourier sine series expansion of $q(x,t)$; because $q(x,t)$ is piecewise smooth, it's valid. To obtain the ODE for $B_n(t)$, multiply both sides by $\sin\frac{p\pi x}{L}$, where p is an integer,

$$\sum_{n=1}^{\infty}\left[B_n'(t)+\frac{kn^2\pi^2}{L^2}B_n(t)\right]\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}=q(x,t)\sin\frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L\sum_{n=1}^{\infty}\left[B_n'(t)+\frac{kn^2\pi^2}{L^2}B_n(t)\right]\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx=\int_0^L q(x,t)\sin\frac{p\pi x}{L}\,dx$$

Split up the integral on the left and bring the constants in front.

$$\sum_{n=1}^{\infty}\left[B_n'(t)+\frac{kn^2\pi^2}{L^2}B_n(t)\right]\int_0^L\sin\frac{n\pi x}{L}\sin\frac{p\pi x}{L}\,dx=\int_0^L q(x,t)\sin\frac{p\pi x}{L}\,dx$$

Since the sine functions are orthogonal, the integral on the left is zero if $n\neq p$. Only if $n=p$ does it yield a nonzero result.

$$\left[B_n'(t)+\frac{kn^2\pi^2}{L^2}B_n(t)\right]\int_0^L\sin^2\frac{n\pi x}{L}\,dx=\int_0^L q(x,t)\sin\frac{n\pi x}{L}\,dx$$

Evaluate the integral on the left.

$$\left[B_n'(t)+\frac{kn^2\pi^2}{L^2}B_n(t)\right]\frac{L}{2}=\int_0^L q(x,t)\sin\frac{n\pi x}{L}\,dx$$

The ODE that $B_n(t)$ satisfies is then

$$B_n'(t)+\frac{kn^2\pi^2}{L^2}B_n(t)=\frac{2}{L}\int_0^L q(x,t)\sin\frac{n\pi x}{L}\,dx,$$

which is a first-order linear inhomogeneous ODE, so it can be solved by using an integrating factor I.

$$I=\exp\left(\int^t\frac{kn^2\pi^2}{L^2}\,ds\right)=\exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

Multiply both sides of the ODE by I.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n'(t)+\frac{kn^2\pi^2}{L^2}\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t)=\left[\frac{2}{L}\int_0^L q(x,t)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

The left side can be written as $d/dt(IB_n)$ by the product rule.

$$\frac{d}{dt}\left[\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t)\right]=\left[\frac{2}{L}\int_0^L q(x,t)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}t\right)$$

Integrate both sides with respect to t.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t)=\int^t\left[\frac{2}{L}\int_0^L q(x,s)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds+C_1$$

The lower limit of integration is arbitrary and can be set to zero. C_1 will be adjusted to account for any choice that's made.

$$\exp\left(\frac{kn^2\pi^2}{L^2}t\right)B_n(t) = \int_0^t \left[\frac{2}{L}\int_0^L q(x,s)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + C_1$$

Solve for $B_n(t)$.

$$B_n(t) = \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\left\{\int_0^t \left[\frac{2}{L}\int_0^L q(x,s)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + C_1\right\}$$

An initial condition is needed to determine C_1. Use equation (1) along with $u(x,0) = f(x)$ to determine it.

$$u(x,0) = \sum_{n=1}^{\infty} B_n(0)\sin\frac{n\pi x}{L} = f(x)$$

The coefficients are known,

$$B_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,$$

so C_1 is as well.

$$B_n(0) = C_1 = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Therefore,

$$\begin{aligned} B_n(t) &= \exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\left\{\int_0^t \left[\frac{2}{L}\int_0^L q(x,s)\sin\frac{n\pi x}{L}\,dx\right]\exp\left(\frac{kn^2\pi^2}{L^2}s\right)ds + \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right\} \\ &= \frac{2}{L}\left\{\int_0^t\int_0^L q(x,s)\sin\frac{n\pi x}{L}\exp\left(\frac{kn^2\pi^2}{L^2}s\right)dx\,ds + \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right\}\exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \end{aligned}$$

and the solution to the PDE is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n(t)\sin\frac{n\pi x}{L} \\ &= \sum_{n=1}^{\infty}\frac{2}{L}\left\{\int_0^t\int_0^L q(x,s)\sin\frac{n\pi x}{L}\exp\left(\frac{kn^2\pi^2}{L^2}s\right)dx\,ds + \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right\}\exp\left(-\frac{kn^2\pi^2}{L^2}t\right)\sin\frac{n\pi x}{L}. \end{aligned}$$