Exercise 3.5.1

Consider

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{3.5.12}$$

(a) Determine b_n from (3.3.11), (3.3.12), and (3.5.6).

(b) For what values of x is (3.5.12) an equality?

(c) Derive the Fourier cosine series for x^3 from (3.5.12). For what values of x will this be an equality?

Solution

Part (a)

Equation (3.3.11) in the text is the Fourier sine series expansion of x (defined on $0 \leq x \leq L$)

$$x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \tag{3.3.11}$$

and equation (3.3.12) in the text is the formula for the coefficients.

$$B_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L}\, dx = -\frac{2(-1)^n L}{n\pi} \tag{3.3.12}$$

Substitute these formulas into equation (3.5.6) in the text to determine the Fourier sine series expansion of x^2.

$$\begin{aligned}
\frac{x^2}{2} &= \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin \frac{\pi x}{L} + \frac{\sin 3\pi x/L}{3^3} + \frac{\sin 5\pi x/L}{5^3} + \cdots\right) \qquad (3.5.6)\\
&= \frac{L}{2}x - \frac{4L^2}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{2}\frac{1-(-1)^n}{n^3} \sin \frac{n\pi x}{L}\\
&= \frac{L}{2}x - \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]L^2}{n^3\pi^3} \sin \frac{n\pi x}{L}\\
&= \frac{L}{2}\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} - \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]L^2}{n^3\pi^3} \sin \frac{n\pi x}{L}\\
&= \frac{L}{2}\sum_{n=1}^{\infty} \left[-\frac{2(-1)^n L}{n\pi}\right] \sin \frac{n\pi x}{L} - \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]L^2}{n^3\pi^3} \sin \frac{n\pi x}{L}\\
&= \sum_{n=1}^{\infty} \left[-\frac{(-1)^n L^2}{n\pi}\right] \sin \frac{n\pi x}{L} - \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]L^2}{n^3\pi^3} \sin \frac{n\pi x}{L}\\
&= \sum_{n=1}^{\infty} \left\{-\frac{(-1)^n L^2}{n\pi} - \frac{2[1-(-1)^n]L^2}{n^3\pi^3}\right\} \sin \frac{n\pi x}{L}
\end{aligned}$$

Multiply both sides by 2.

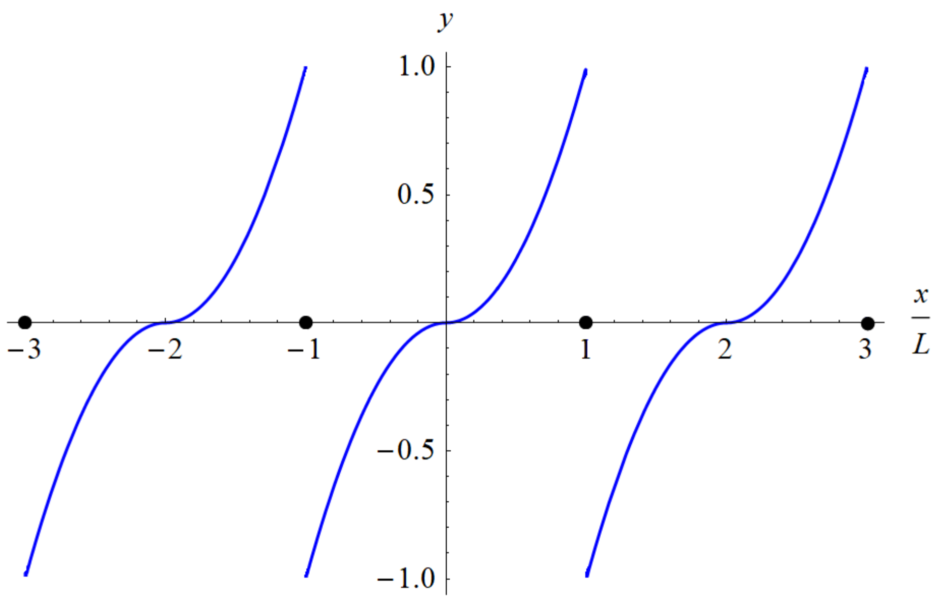
$$x^2 = \sum_{n=1}^{\infty} \left\{-\frac{2(-1)^n L^2}{n\pi} - \frac{4[1-(-1)^n]L^2}{n^3\pi^3}\right\} \sin \frac{n\pi x}{L}$$

Part (b)

Divide both sides by L^2.

$$\left(\frac{x}{L}\right)^2 = \sum_{n=1}^{\infty}\left\{-\frac{2(-1)^n}{n\pi} - \frac{4[1-(-1)^n]}{n^3\pi^3}\right\}\sin\frac{n\pi x}{L}$$

Below is a graph of the right side versus x/L.



Note that while x is only defined on $0 \leq x \leq L$, the sine series is defined on the whole line. There are points of discontinuity at odd multiples of L: $x = (2n+1)L$, where n is any integer. It's continuous everywhere else; in other words, there's equality in equation (3.5.12) at $0 \leq x < L$.

Part (c)

Start with the result from part (a).

$$x^2 = \sum_{n=1}^{\infty}\left\{-\frac{2(-1)^n L^2}{n\pi} - \frac{4[1-(-1)^n]L^2}{n^3\pi^3}\right\}\sin\frac{n\pi x}{L}$$

Integrate both sides with respect to x.

$$\begin{aligned}
\int x^2\,dx &= \int \sum_{n=1}^{\infty}\left\{-\frac{2(-1)^n L^2}{n\pi} - \frac{4[1-(-1)^n]L^2}{n^3\pi^3}\right\}\sin\frac{n\pi x}{L}\,dx \\
\frac{x^3}{3} &= \sum_{n=1}^{\infty}\left\{-\frac{2(-1)^n L^2}{n\pi} - \frac{4[1-(-1)^n]L^2}{n^3\pi^3}\right\}\int \sin\frac{n\pi x}{L}\,dx + C \\
&= \sum_{n=1}^{\infty}\left\{-\frac{2(-1)^n L^2}{n\pi} - \frac{4[1-(-1)^n]L^2}{n^3\pi^3}\right\}\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L} + C \\
&= \sum_{n=1}^{\infty}\left\{\frac{2(-1)^n L^3}{n^2\pi^2} + \frac{4[1-(-1)^n]L^3}{n^4\pi^4}\right\}\cos\frac{n\pi x}{L} + C
\end{aligned}$$

In order to determine C, integrate both sides with respect to x from 0 to L.

$$\int_0^L \frac{x^3}{3}\,dx = \int_0^L \left\{\sum_{n=1}^{\infty}\left\{\frac{2(-1)^n L^3}{n^2\pi^2} + \frac{4[1-(-1)^n]L^3}{n^4\pi^4}\right\}\cos\frac{n\pi x}{L} + C\right\} dx$$

$$\frac{L^4}{12} = \sum_{n=1}^{\infty}\left\{\frac{2(-1)^n L^3}{n^2\pi^2} + \frac{4[1-(-1)^n]L^3}{n^4\pi^4}\right\}\underbrace{\int_0^L \cos\frac{n\pi x}{L}\,dx}_{=\,0} + C\int_0^L dx$$

$$= CL$$

Solve for C.

$$C = \frac{L^3}{12}$$

Substitute this formula into the one for $x^3/3$.

$$\frac{x^3}{3} = \sum_{n=1}^{\infty}\left\{\frac{2(-1)^n L^3}{n^2\pi^2} + \frac{4[1-(-1)^n]L^3}{n^4\pi^4}\right\}\cos\frac{n\pi x}{L} + \frac{L^3}{12}$$

Multiply both sides by 3.

$$x^3 = \sum_{n=1}^{\infty}\left\{\frac{6(-1)^n L^3}{n^2\pi^2} + \frac{12[1-(-1)^n]L^3}{n^4\pi^4}\right\}\cos\frac{n\pi x}{L} + \frac{L^3}{4}$$

Divide both sides by L^3.

$$\left(\frac{x}{L}\right)^3 = \sum_{n=1}^{\infty}\left\{\frac{6(-1)^n}{n^2\pi^2} + \frac{12[1-(-1)^n]}{n^4\pi^4}\right\}\cos\frac{n\pi x}{L} + \frac{1}{4}$$

Below is a graph of the right side versus x/L. The graph is continuous everywhere, so there's equality for $0 \le x \le L$.