Exercise 3.5.3

Generalize Exercise 3.5.2, in order to derive the Fourier sine series of x^m, m odd.

Solution

In Exercise 3.5.2 the following series expansions were found.

$$x = \sum_{n=1}^{\infty}\left[-\frac{2(-1)^n L}{n\pi}\right]\sin\frac{n\pi x}{L}$$

$$x^2 = \sum_{n=1}^{\infty}\frac{2\cdot 2(-1)^n L^2}{n^2\pi^2}\cos\frac{n\pi x}{L} + \frac{L^2}{3}$$

$$x^3 = \sum_{n=1}^{\infty}\left[\frac{3\cdot 2\cdot 2(-1)^n L^3}{n^3\pi^3} - \frac{3\cdot 2(-1)^n L^3}{3n\pi}\right]\sin\frac{n\pi x}{L}$$

Integrate both sides of this expansion for x^3 with respect to x.

$$\begin{aligned}\frac{x^4}{4} &= \sum_{n=1}^{\infty}\left[\frac{3\cdot 2\cdot 2(-1)^n L^3}{n^3\pi^3} - \frac{3\cdot 2(-1)^n L^3}{3n\pi}\right]\left(-\frac{L}{n\pi}\right)\cos\frac{n\pi x}{L} + C_1 \\ &= \sum_{n=1}^{\infty}\left[-\frac{3\cdot 2\cdot 2(-1)^n L^4}{n^4\pi^4} + \frac{3\cdot 2(-1)^n L^4}{3n^2\pi^2}\right]\cos\frac{n\pi x}{L} + C_1\end{aligned}$$

Integrate both sides with respect to x from 0 to L.

$$\frac{L^5}{5\cdot 4} = C_1 L \quad\rightarrow\quad C_1 = \frac{L^4}{5\cdot 4}$$

Substitute this value for C_1 back into the formula for $x^4/4$.

$$\frac{x^4}{4} = \sum_{n=1}^{\infty}\left[-\frac{3\cdot 2\cdot 2(-1)^n L^4}{n^4\pi^4} + \frac{3\cdot 2(-1)^n L^4}{3n^2\pi^2}\right]\cos\frac{n\pi x}{L} + \frac{L^4}{5\cdot 4}$$

Multiply both sides by 4.

$$x^4 = \sum_{n=1}^{\infty}\left[-\frac{4\cdot 3\cdot 2\cdot 2(-1)^n L^4}{n^4\pi^4} + \frac{4\cdot 3\cdot 2(-1)^n L^4}{3n^2\pi^2}\right]\cos\frac{n\pi x}{L} + \frac{4L^4}{5\cdot 4}$$

Integrate both sides with respect to x.

$$\begin{aligned}\frac{x^5}{5} &= \sum_{n=1}^{\infty}\left[-\frac{4\cdot 3\cdot 2\cdot 2(-1)^n L^4}{n^4\pi^4} + \frac{4\cdot 3\cdot 2(-1)^n L^4}{3n^2\pi^2}\right]\left(\frac{L}{n\pi}\right)\sin\frac{n\pi x}{L} + \frac{4L^4}{5\cdot 4}x + C_2 \\ &= \sum_{n=1}^{\infty}\left[-\frac{4\cdot 3\cdot 2\cdot 2(-1)^n L^5}{n^5\pi^5} + \frac{4\cdot 3\cdot 2(-1)^n L^5}{3n^3\pi^3}\right]\sin\frac{n\pi x}{L} + \frac{4L^4}{5\cdot 4}\sum_{n=1}^{\infty}\left[-\frac{2(-1)^n L}{n\pi}\right]\sin\frac{n\pi x}{L} + C_2 \\ &= \sum_{n=1}^{\infty}\left[-\frac{4\cdot 3\cdot 2\cdot 2(-1)^n L^5}{n^5\pi^5} + \frac{4\cdot 3\cdot 2(-1)^n L^5}{3n^3\pi^3} - \frac{4\cdot 2(-1)^n L^5}{5\cdot 4n\pi}\right]\sin\frac{n\pi x}{L} + C_2\end{aligned}$$

Set $x = 0$ in the equation to determine C_2.

$$0 = C_2$$

The formula for $x^5/5$ then becomes

$$\frac{x^5}{5} = \sum_{n=1}^{\infty} \left[-\frac{4 \cdot 3 \cdot 2 \cdot 2(-1)^n L^5}{n^5 \pi^5} + \frac{4 \cdot 3 \cdot 2(-1)^n L^5}{3n^3\pi^3} - \frac{4 \cdot 2(-1)^n L^5}{5 \cdot 4n\pi} \right] \sin \frac{n\pi x}{L}.$$

Multiply both sides by 5.

$$\begin{aligned}
x^5 &= \sum_{n=1}^{\infty} \left[-\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2(-1)^n L^5}{n^5 \pi^5} + \frac{5 \cdot 4 \cdot 3 \cdot 2(-1)^n L^5}{3n^3\pi^3} - \frac{5 \cdot 4 \cdot 3 \cdot 2(-1)^n L^5}{5 \cdot 4 \cdot 3n\pi} \right] \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left[-\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2(-1)^n L^5}{n^5 \pi^5} + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2(-1)^n L^5}{3 \cdot 2n^3\pi^3} - \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2(-1)^n L^5}{5 \cdot 4 \cdot 3 \cdot 2n\pi} \right] \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left[-\frac{5! \cdot 2(-1)^n L^5}{n^5 \pi^5} + \frac{5! \cdot 2(-1)^n L^5}{3! n^3\pi^3} - \frac{5! \cdot 2(-1)^n L^5}{5! n\pi} \right] \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} 5! \left(-\frac{1}{n^5\pi^5} + \frac{1}{3! n^3 \pi^3} - \frac{1}{5! n\pi} \right) 2(-1)^n L^5 \sin \frac{n\pi x}{L}
\end{aligned}$$

The expansion for x^3 can be written similarly as

$$\begin{aligned}
x^3 &= \sum_{n=1}^{\infty} \left[\frac{3 \cdot 2 \cdot 2(-1)^n L^3}{n^3\pi^3} - \frac{3 \cdot 2(-1)^n L^3}{3n\pi} \right] \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left[\frac{3 \cdot 2 \cdot 2(-1)^n L^3}{n^3\pi^3} - \frac{3 \cdot 2 \cdot 2(-1)^n L^3}{3 \cdot 2n\pi} \right] \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left[\frac{3! \cdot 2(-1)^n L^3}{n^3\pi^3} - \frac{3! \cdot 2(-1)^n L^3}{3! n\pi} \right] \sin \frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} 3! \left(\frac{1}{n^3\pi^3} - \frac{1}{3! n\pi} \right) 2(-1)^n L^3 \sin \frac{n\pi x}{L}.
\end{aligned}$$

Using these formulas for x^3 and x^5, a general formula for x^m can be deduced, where m is odd.

$$x^m = \sum_{n=1}^{\infty} m! \left[\sum_{i=1}^{m} \cos\left(\frac{i+1}{2}\pi\right) \frac{1-(-1)^i}{2(1+m-i)!(n\pi)^i} \right] 2(-1)^n L^m \sin \frac{n\pi x}{L}$$

The cosine function takes care of the alternating signs, the $[1-(-1)^i]/2$ term makes it so that only odd powers are present, and $(1+m-i)!$ takes care of the coefficient of $(n\pi)^i$. Notice that the summand is zero if i is even; this sum can be simplified then (that is, made to converge faster) by summing over the odd integers only.

Set $i = 2k - 1$ and simplify the resulting formula.

$$\begin{aligned}
x^m &= \sum_{n=1}^{\infty} m! \left\{ \sum_{2k-1=1}^{m} \cos\left[\frac{(2k-1)+1}{2}\pi\right] \frac{2}{2[1+m-(2k-1)]!(n\pi)^{2k-1}} \right\} 2(-1)^n L^m \sin\frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} m! \left\{ \sum_{2k=2}^{m+1} \cos\left(\frac{2k}{2}\pi\right) \frac{1}{(2+m-2k)!(n\pi)^{2k-1}} \right\} 2(-1)^n L^m \sin\frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} m! \left\{ \sum_{k=1}^{\frac{m+1}{2}} \cos(k\pi) \frac{1}{[m-2(k-1)]!(n\pi)^{2k-1}} \right\} 2(-1)^n L^m \sin\frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} m! \left\{ \sum_{k=1}^{\frac{m+1}{2}} (-1)^k \frac{1}{[m-2(k-1)]!(n\pi)^{2k-1}} \right\} 2(-1)^n L^m \sin\frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{\frac{m+1}{2}} \frac{2(-1)^{k+n} m! L^m}{[m-2(k-1)]!(n\pi)^{2k-1}} \right\} \sin\frac{n\pi x}{L}
\end{aligned}$$

This is the Fourier sine series expansion of x^m, where m is odd.