Exercise 3.5.4

Suppose that $\cosh x \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$.

(a) Determine $b_n$ by correctly differentiating this series twice.

(b) Determine $b_n$ by integrating this series twice.

Solution

Part (a)

Consider the Fourier sine series expansion of $\cosh x$ (defined on $0 \leq x \leq L$).

$$\cosh x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Differentiate both sides with respect to $x$. Although hyperbolic cosine is a continuous function, $\cosh 0 \neq 0$ and $\cosh L \neq 0$, so the sine series cannot be differentiated with respect to $x$ term by term. The right side is expected to be a cosine series.

$$\frac{d}{dx} \left( \cosh x \right) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \tag{1}$$

To get $A_0$, integrate both sides with respect to $x$ from 0 to $L$.

$$\int_0^L \frac{d}{dx} \left( \cosh x \right) \, dx = \int_0^L \left( A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \right) \, dx$$

$$= A_0 \int_0^L \, dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \, dx$$

$$= A_0 L$$

Solve for $A_0$.

$$A_0 = \frac{1}{L} \int_0^L \frac{d}{dx} \left( \cosh x \right) \, dx$$

$$= \frac{1}{L} \left( \cosh L - 1 \right)$$

To get $A_n$, multiply both sides of equation (1) by $\cos \frac{p\pi x}{L}$, where $p$ is an integer,

$$\frac{d}{dx} \left( \cosh x \right) \cos \frac{p\pi x}{L} = A_0 \cos \frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}$$

and then integrate both sides with respect to $x$ from 0 to $L$.

$$\int_0^L \frac{d}{dx} \left( \cosh x \right) \cos \frac{p\pi x}{L} \, dx = \int_0^L \left( A_0 \cos \frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L} \right) \, dx$$

$$= A_0 \int_0^L \cos \frac{p\pi x}{L} \, dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L} \, dx$$
Since the cosine functions are orthogonal, this second integral on the right is zero if \( n \neq p \). Only if \( n = p \) is it nonzero.

\[
\int_0^L \frac{d}{dx}(\cosh x) \cos \frac{n\pi x}{L} \, dx = A_n \int_0^L \cos^2 \frac{n\pi x}{L} \, dx = A_n \left( \frac{L}{2} \right)
\]

Solve for \( A_n \).

\[
A_n = \frac{2}{L} \int_0^L \frac{d}{dx}(\cosh x) \cos \frac{n\pi x}{L} \, dx
= \frac{2}{L} \int_0^L \sinh x \cos \frac{n\pi x}{L} \, dx
= \frac{2}{L} \int_0^L \frac{e^x - e^{-x}}{2} \cos \frac{n\pi x}{L} \, dx
= \frac{2L[-1 + (-1)^n \cosh L]}{n^2\pi^2 + L^2}
\]

As a result, equation (1) becomes

\[
\sinh x = \frac{1}{L}(\cosh L - 1) + \sum_{n=1}^{\infty} \frac{2L[-1 + (-1)^n \cosh L]}{n^2\pi^2 + L^2} \cos \frac{n\pi x}{L}.
\]

Differentiate both sides with respect to \( x \) once more. Because hyperbolic sine is continuous, the cosine series can be differentiated with respect to \( x \) term by term.

\[
\frac{d}{dx}(\sinh x) = \sum_{n=1}^{\infty} \frac{2L[-1 + (-1)^n \cosh L]}{n^2\pi^2 + L^2} \left( -\frac{n\pi}{L} \right) \sin \frac{n\pi x}{L}
\]

Simplify both sides.

\[
\cosh x = \sum_{n=1}^{\infty} \frac{2n\pi[1 - (-1)^n \cosh L]}{n^2\pi^2 + L^2} \sin \frac{n\pi x}{L}
\]

Set the coefficients equal to each other.

\[
b_n = \frac{2n\pi[1 - (-1)^n \cosh L]}{n^2\pi^2 + L^2}
\]
Part (b)

Consider the Fourier sine series expansion of $\cosh x$ (defined on $0 \leq x \leq L$).

$$\cosh x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Integrate both sides with respect to $x$.

$$\sinh x = \int \cosh x \, dx = \int \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \, dx + C_1$$

$$= \sum_{n=1}^{\infty} b_n \int \sin \frac{n\pi x}{L} \, dx + C_1$$

$$= \sum_{n=1}^{\infty} b_n \left( -\frac{L}{n\pi} \right) \cos \frac{n\pi x}{L} + C_1$$

To determine $C_1$, integrate both sides with respect to $x$ from 0 to $L$.

$$\int_0^L \sinh x \, dx = \int_0^L \sum_{n=1}^{\infty} b_n \left( -\frac{L}{n\pi} \right) \cos \frac{n\pi x}{L} + C_1 \, dx$$

$$= \sum_{n=1}^{\infty} b_n \left( -\frac{L}{n\pi} \right) \left[ \cos \frac{n\pi x}{L} \right]_0^L + C_1 \int_0^L \, dx$$

$$= C_1 L$$

Solve for $C_1$.

$$C_1 = \frac{1}{L} \int_0^L \sinh x \, dx$$

$$= \frac{\cosh L - 1}{L}$$

As a result, the formula for $\sinh x$ becomes

$$\sinh x = \sum_{n=1}^{\infty} b_n \left( -\frac{L}{n\pi} \right) \cos \frac{n\pi x}{L} + \frac{\cosh L - 1}{L}.$$
This equation holds for every value of $x$, so set $x = 0$ to determine $C_2$.

$$1 = C_2$$

As a result,

$$\cosh x = \sum_{n=1}^{\infty} b_n \left( -\frac{L}{n\pi} \right) \left( \frac{L}{n\pi} \right) \sin \frac{n\pi x}{L} + \cosh L - 1 \sum_{n=1}^{\infty} \left[ -\frac{2(-1)^n L}{n\pi} \right] \sin \frac{n\pi x}{L} + \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi} \sin \frac{n\pi x}{L}$$

Substitute the Fourier sine series expansions of 1 and $x$ and simplify the right side.

$$\cosh x = \sum_{n=1}^{\infty} b_n \left( -\frac{L}{n\pi} \right) \left( \frac{L}{n\pi} \right) \sin \frac{n\pi x}{L} + \cosh L - 1 \sum_{n=1}^{\infty} \left[ -\frac{2(-1)^n L}{n\pi} \right] \sin \frac{n\pi x}{L} + \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{n\pi} \sin \frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} \left\{ -\frac{L^2}{n^2\pi^2} b_n - \frac{2(-1)^n (\cosh L - 1)}{n\pi} + \frac{2[1 - (-1)^n]}{n\pi} \right\} \sin \frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} \left\{ -\frac{L^2}{n^2\pi^2} b_n + \frac{2[1 - (-1)^n \cosh L]}{n\pi} \right\} \sin \frac{n\pi x}{L}$$

Set the coefficients equal to each other.

$$b_n = -\frac{L^2}{n^2\pi^2} b_n + \frac{2[1 - (-1)^n \cosh L]}{n\pi}$$

Solve for $b_n$.

$$b_n \left( 1 + \frac{L^2}{n^2\pi^2} \right) = \frac{2[1 - (-1)^n \cosh L]}{n\pi}$$

$$b_n \left( \frac{n^2\pi^2 + L^2}{n^2\pi^2} \right) = \frac{2[1 - (-1)^n \cosh L]}{n\pi}$$

$$b_n = \frac{2n\pi[1 - (-1)^n \cosh L]}{n^2\pi^2 + L^2}$$

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