Exercise 3.5.4

Suppose that $\cosh x \sim \sum_{n=1}^{\infty} b_n \sin n\pi x/L$.

(a) Determine b_n by correctly differentiating this series twice.

(b) Determine b_n by integrating this series twice.

Solution

Part (a)

Consider the Fourier sine series expansion of $\cosh x$ (defined on $0 \leq x \leq L$).

$$\cosh x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Differentiate both sides with respect to x. Although hyperbolic cosine is a continuous function, $\cosh 0 \neq 0$ and $\cosh L \neq 0$, so the sine series cannot be differentiated with respect to x term by term. The right side is expected to be a cosine series.

$$\frac{d}{dx}(\cosh x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \tag{1}$$

To get A_0, integrate both sides with respect to x from 0 to L.

$$\begin{aligned}\int_0^L \frac{d}{dx}(\cosh x)\,dx &= \int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}\right) dx \\ &= A_0 \int_0^L dx + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^L \cos \frac{n\pi x}{L}\,dx}_{=0} \\ &= A_0 L\end{aligned}$$

Solve for A_0.

$$\begin{aligned}A_0 &= \frac{1}{L}\int_0^L \frac{d}{dx}(\cosh x)\,dx \\ &= \frac{1}{L}(\cosh L - 1)\end{aligned}$$

To get A_n, multiply both sides of equation (1) by $\cos \frac{p\pi x}{L}$, where p is an integer,

$$\frac{d}{dx}(\cosh x)\cos \frac{p\pi x}{L} = A_0 \cos \frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\begin{aligned}\int_0^L \frac{d}{dx}(\cosh x)\cos \frac{p\pi x}{L}\,dx &= \int_0^L \left(A_0 \cos \frac{p\pi x}{L} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}\right) dx \\ &= A_0 \underbrace{\int_0^L \cos \frac{p\pi x}{L}\,dx}_{=0} + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{p\pi x}{L}\,dx\end{aligned}$$

Since the cosine functions are orthogonal, this second integral on the right is zero if $n \neq p$. Only if $n = p$ is it nonzero.

$$\begin{aligned}\int_0^L \frac{d}{dx}(\cosh x)\cos\frac{n\pi x}{L}\,dx &= A_n \int_0^L \cos^2\frac{n\pi x}{L}\,dx \\ &= A_n\left(\frac{L}{2}\right)\end{aligned}$$

Solve for A_n.

$$\begin{aligned}A_n &= \frac{2}{L}\int_0^L \frac{d}{dx}(\cosh x)\cos\frac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\int_0^L \sinh x\cos\frac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\int_0^L \frac{e^x - e^{-x}}{2}\cos\frac{n\pi x}{L}\,dx \\ &= \frac{2L[-1 + (-1)^n\cosh L]}{n^2\pi^2 + L^2}\end{aligned}$$

As a result, equation (1) becomes

$$\sinh x = \frac{1}{L}(\cosh L - 1) + \sum_{n=1}^{\infty}\frac{2L[-1 + (-1)^n\cosh L]}{n^2\pi^2 + L^2}\cos\frac{n\pi x}{L}.$$

Differentiate both sides with respect to x once more. Because hyperbolic sine is continuous, the cosine series can be differentiated with respect to x term by term.

$$\frac{d}{dx}(\sinh x) = \sum_{n=1}^{\infty}\frac{2L[-1 + (-1)^n\cosh L]}{n^2\pi^2 + L^2}\left(-\frac{n\pi}{L}\right)\sin\frac{n\pi x}{L}$$

Simplify both sides.

$$\cosh x = \sum_{n=1}^{\infty}\frac{2n\pi[1 - (-1)^n\cosh L]}{n^2\pi^2 + L^2}\sin\frac{n\pi x}{L}$$

Set the coefficients equal to each other.

$$b_n = \frac{2n\pi[1 - (-1)^n\cosh L]}{n^2\pi^2 + L^2}$$

Part (b)

Consider the Fourier sine series expansion of $\cosh x$ (defined on $0 \leq x \leq L$).

$$\cosh x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

Integrate both sides with respect to x.

$$\begin{aligned}\sinh x &= \int \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}\, dx + C_1 \\ &= \sum_{n=1}^{\infty} b_n \int \sin \frac{n\pi x}{L}\, dx + C_1 \\ &= \sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi}\right) \cos \frac{n\pi x}{L} + C_1\end{aligned}$$

To determine C_1, integrate both sides with respect to x from 0 to L.

$$\begin{aligned}\int_0^L \sinh x\, dx &= \int_0^L \left[\sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi}\right) \cos \frac{n\pi x}{L} + C_1\right] dx \\ &= \sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi}\right) \underbrace{\int_0^L \cos \frac{n\pi x}{L}\, dx}_{=\,0} + C_1 \int_0^L dx \\ &= C_1 L\end{aligned}$$

Solve for C_1.

$$\begin{aligned}C_1 &= \frac{1}{L}\int_0^L \sinh x\, dx \\ &= \frac{\cosh L - 1}{L}\end{aligned}$$

As a result, the formula for $\sinh x$ becomes

$$\sinh x = \sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi}\right) \cos \frac{n\pi x}{L} + \frac{\cosh L - 1}{L}.$$

Integrate both sides with respect to x once more.

$$\begin{aligned}\cosh x &= \int \left[\sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi}\right) \cos \frac{n\pi x}{L} + \frac{\cosh L - 1}{L}\right] dx + C_2 \\ &= \sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi}\right) \int \cos \frac{n\pi x}{L}\, dx + \int \frac{\cosh L - 1}{L}\, dx + C_2 \\ &= \sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi}\right)\left(\frac{L}{n\pi}\right) \sin \frac{n\pi x}{L} + \frac{\cosh L - 1}{L} x + C_2\end{aligned}$$

This equation holds for every value of x, so set $x = 0$ to determine C_2.

$$1 = C_2$$

As a result,

$$\cosh x = \sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi}\right)\left(\frac{L}{n\pi}\right) \sin\frac{n\pi x}{L} + \frac{\cosh L - 1}{L} x + 1.$$

Substitute the Fourier sine series expansions of 1 and x and simplify the right side.

$$\begin{aligned}
\cosh x &= \sum_{n=1}^{\infty} b_n \left(-\frac{L}{n\pi}\right)\left(\frac{L}{n\pi}\right) \sin\frac{n\pi x}{L} + \frac{\cosh L - 1}{L} \sum_{n=1}^{\infty} \left[-\frac{2(-1)^n L}{n\pi}\right] \sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} \sin\frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left(-\frac{L^2}{n^2\pi^2} b_n\right) \sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \left[-\frac{2(-1)^n(\cosh L - 1)}{n\pi}\right] \sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} \sin\frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left\{-\frac{L^2}{n^2\pi^2} b_n - \frac{2(-1)^n(\cosh L - 1)}{n\pi} + \frac{2[1-(-1)^n]}{n\pi}\right\} \sin\frac{n\pi x}{L} \\
&= \sum_{n=1}^{\infty} \left\{-\frac{L^2}{n^2\pi^2} b_n + \frac{2[1-(-1)^n \cosh L]}{n\pi}\right\} \sin\frac{n\pi x}{L}
\end{aligned}$$

Set the coefficients equal to each other.

$$b_n = -\frac{L^2}{n^2\pi^2} b_n + \frac{2[1-(-1)^n \cosh L]}{n\pi}$$

Solve for b_n.

$$\begin{aligned}
b_n \left(1 + \frac{L^2}{n^2\pi^2}\right) &= \frac{2[1-(-1)^n \cosh L]}{n\pi} \\
b_n \left(\frac{n^2\pi^2 + L^2}{n^2\pi^2}\right) &= \frac{2[1-(-1)^n \cosh L]}{n\pi} \\
b_n &= \frac{2n\pi[1-(-1)^n \cosh L]}{n^2\pi^2 + L^2}
\end{aligned}$$