Problem 12-15

A particle is moving with a velocity of v_0 when $s = 0$ and $t = 0$. If it is subjected to a deceleration of $a = -kv^3$, where k is a constant, determine its velocity and position as functions of time.

Solution

The acceleration and velocity are related by

$$a = \frac{dv}{dt} = -kv^3.$$

Divide both sides by v^3.

$$\frac{\frac{dv}{dt}}{v^3} = -k$$

Rewrite the left side.

$$\frac{d}{dt}\left(-\frac{1}{2v^2}\right) = -k$$

Integrate both sides with respect to t.

$$-\frac{1}{2v^2} = -kt + C_1 \tag{1}$$

Use the fact that the velocity is v_0 at $t = 0$ to determine C_1.

$$-\frac{1}{2v_0^2} = C_1$$

As a result, equation (1) becomes

$$-\frac{1}{2v^2} = -kt - \frac{1}{2v_0^2}.$$

Multiply both sides by -1 and invert both sides.

$$2v^2 = \frac{1}{kt + \frac{1}{2v_0^2}} = \frac{2v_0^2}{2kv_0^2 t + 1}$$

Solve for v.

$$\boxed{v(t) = \frac{v_0}{\sqrt{2kv_0^2 t + 1}}}$$

Velocity is related to position by

$$v = \frac{ds}{dt} = \frac{v_0}{\sqrt{2kv_0^2 t + 1}}.$$

Integrate both sides with respect to t to get $s(t)$.

$$s(t) = \int^t \frac{v_0}{\sqrt{2kv_0^2 t' + 1}}\, dt'$$

Make the following substitution.

$$u = 2kv_0^2 t' + 1$$

$$du = 2kv_0^2\, dt' \quad \rightarrow \quad \frac{du}{2kv_0^2} = dt'$$

The position is then

$$s(t) = \int^{2kv_0^2 t+1} \frac{v_0}{\sqrt{u}} \left(\frac{du}{2kv_0^2}\right)$$

$$= \frac{1}{2kv_0} \int^{2kv_0^2 t+1} u^{-1/2}\, du$$

$$= \frac{1}{2kv_0}(2u^{1/2})\Big|^{2kv_0^2 t+1} + C_2$$

$$= \frac{\sqrt{2kv_0^2 t+1}}{kv_0} + C_2.$$

Use the fact that $s = 0$ at $t = 0$ to determine C_2.

$$s(0) = \frac{1}{kv_0} + C_2 = 0 \quad \rightarrow \quad C_2 = -\frac{1}{kv_0}$$

Therefore, the particle's position is

$$\boxed{s(t) = \frac{\sqrt{2kv_0^2 t+1}}{kv_0} - \frac{1}{kv_0}.}$$