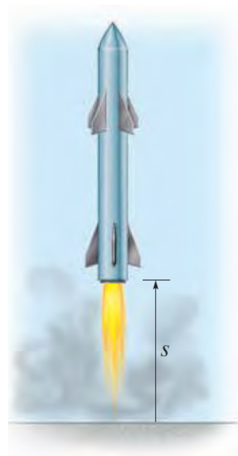
Problem 12-18

The acceleration of a rocket traveling upward is given by $a = (6 + 0.02s)$ m/s^2, where s is in meters. Determine the time needed for the rocket to reach an altitude of $s = 100$ m. Initially, $v = 0$ and $s = 0$ when $t = 0$.



Prob. 12–18

Solution

The acceleration and position are related by

$$a = \frac{d^2 s}{dt^2} = 6 + 0.02s,$$

so the initial value problem to solve here is

$$\frac{d^2 s}{dt^2} - 0.02s = 6, \quad s(0) = 0, \ \frac{ds}{dt}(0) = 0.$$

This is a linear inhomogeneous ODE, so the general solution can be written as the sum of a complementary solution and a particular solution.

$$s = s_c + s_p$$

The complementary solution satisfies the associated homogeneous ODE.

$$\frac{d^2 s_c}{dt^2} - 0.02s_c = 0$$

Its general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$s_c = C_1 \cosh(\sqrt{0.02}t) + C_2 \sinh(\sqrt{0.02}t)$$

Since the inhomogeneous term in the ODE for s is a constant, the particular solution is a constant as well.

$$\cancel{\frac{d^2 s_p}{dt^2}} - 0.02s_p = 6 \quad \rightarrow \quad s_p = -\frac{6}{0.02} = -300$$

The general solution for s is then

$$\begin{aligned} s(t) &= s_c(t) + s_p(t) \\ &= C_1 \cosh(\sqrt{0.02}t) + C_2 \sinh(\sqrt{0.02}t) - 300. \end{aligned}$$

Differentiate it with respect to t.

$$\frac{ds}{dt}(t) = C_1\sqrt{0.02}\sinh(\sqrt{0.02}t) + C_2\sqrt{0.02}\cosh(\sqrt{0.02}t)$$

Apply the initial conditions now to determine C_1 and C_2.

$$s(0) = C_1 - 300 = 0$$

$$\frac{ds}{dt}(0) = C_2\sqrt{0.02} = 0$$

Solving this system of equations yields $C_1 = 300$ and $C_2 = 0$. Therefore,

$$s(t) = 300\cosh(\sqrt{0.02}t) - 300.$$

To determine the time needed for the rocket to reach an altitude of $s = 100$ m, set $s(t) = 100$ and solve for t.

$$300\cosh(\sqrt{0.02}t) - 300 = 100$$

$$\cosh(\sqrt{0.02}t) = \frac{4}{3}$$

$$\sqrt{0.02}t = \cosh^{-1}\left(\frac{4}{3}\right)$$

$$t = \frac{1}{\sqrt{0.02}}\cosh^{-1}\left(\frac{4}{3}\right) \text{ s} \approx 5.62 \text{ s}$$