Problem 12-25

If the effects of atmospheric resistance are accounted for, a freely falling body has an acceleration defined by the equation $a = 9.81[1 - v^2(10^{-4})]$ m/s^2, where v is in m/s and the positive direction is downward. If the body is released from rest at a *very high altitude*, determine (a) the velocity when $t = 5$ s, and (b) the body's terminal or maximum attainable velocity (as $t \to \infty$).

Solution

The acceleration and velocity are related by

$$a = \frac{dv}{dt} = 9.81[1 - v^2(10^{-4})]. \tag{1}$$

Once the maximum velocity is reached, the velocity stops increasing.

$$0 = 9.81[1 - v_\infty^2(10^{-4})]$$

Solve for v_∞, the terminal velocity.

$$v_\infty = 100\ \frac{\text{m}}{\text{s}}$$

Solve for $v(t)$ now by separating variables in equation (1).

$$\frac{dv}{1 - v^2(10^{-4})} = 9.81\, dt$$

Integrate both sides.

$$\int^v \frac{dv'}{1 - v'^2(10^{-4})} = \int^t 9.81\, dt'$$

Make the following substitution in the integral on the left.

$$v' = 100 \sin\theta$$
$$dv' = 100 \cos\theta\, d\theta$$

Evaluate the integral on the right.

$$\int^{\sin^{-1}\left(\frac{v}{100}\right)} \frac{100\cos\theta\, d\theta}{1 - \sin^2\theta} = 9.81t + C$$

$$\int^{\sin^{-1}\left(\frac{v}{100}\right)} \frac{100\cos\theta\, d\theta}{\cos^2\theta} = 9.81t + C$$

$$100 \int^{\sin^{-1}\left(\frac{v}{100}\right)} \frac{d\theta}{\cos\theta} = 9.81t + C$$

$$100 \int^{\sin^{-1}\left(\frac{v}{100}\right)} \sec\theta\, d\theta = 9.81t + C$$

$$100 \ln|\sec\theta + \tan\theta| \Bigg|^{\sin^{-1}\left(\frac{v}{100}\right)} = 9.81t + C$$

$$100 \ln\left|\sec\sin^{-1}\frac{v}{100} + \tan\sin^{-1}\frac{v}{100}\right| = 9.81t + C \tag{2}$$

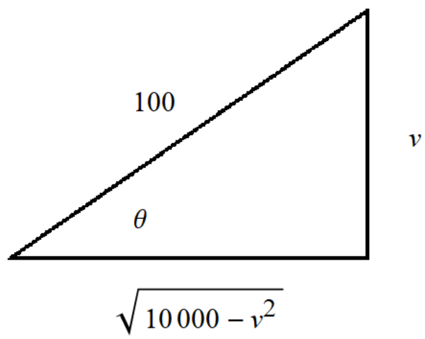
Use the fact that the body is released from rest to determine C.

$$100\ln\left|\sec\sin^{-1}\frac{0}{100}+\tan\sin^{-1}\frac{0}{100}\right| = 9.81(0)+C \quad \rightarrow \quad C = 100\ln 1 = 0$$

As a result, equation (2) becomes

$$100\ln\left|\sec\sin^{-1}\frac{v}{100}+\tan\sin^{-1}\frac{v}{100}\right| = 9.81t. \tag{3}$$

Draw the implied right triangle to determine the secant and tangent.



Use it to simplify equation (3) and then solve for v.

$$100\ln\left|\frac{100}{\sqrt{10\,000-v^2}}+\frac{v}{\sqrt{10\,000-v^2}}\right| = 9.81t$$

$$\ln\left|\frac{100+v}{\sqrt{10\,000-v^2}}\right| = 0.0981t$$

$$\left|\frac{100+v}{\sqrt{10\,000-v^2}}\right| = e^{0.0981t}$$

$$\left(\frac{100+v}{\sqrt{10\,000-v^2}}\right)^2 = (e^{0.0981t})^2$$

$$\frac{10\,000+200v+v^2}{10\,000-v^2} = e^{0.1962t}$$

$$10\,000+200v+v^2 = 10\,000e^{0.1962t}-e^{0.1962t}v^2$$

$$(1+e^{0.1962t})v^2+200v+10\,000(1-e^{0.1962t}) = 0$$

$$v = \frac{-200\pm\sqrt{40\,000-4\cdot 10\,000(1-e^{0.1962t})(1+e^{0.1962t})}}{2(1+e^{0.1962t})}$$

Choose the positive sign since the body moves downward in the positive direction.

$$v(t) = \frac{-100+100\sqrt{e^{0.3924t}}}{1+e^{0.1962t}} = 100\frac{e^{0.1962t}-1}{e^{0.1962t}+1} \quad \Rightarrow \quad v(5)\approx 45.5\ \frac{\text{m}}{\text{s}}$$