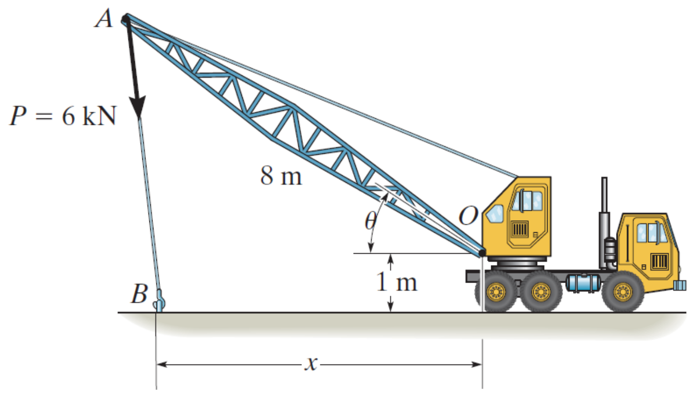
Problem 4-12

The towline exerts a force of $P = 6$ kN at the end of the 8-m-long crane boom. If $x = 10$ m, determine the position θ of the boom so that this force creates a maximum moment about point O. What is this moment?



Probs. 4–11/12

Solution

Treat O as the origin of an xyz-coordinate system and write the position vectors to points A and B.

$$\mathbf{r}_A = 8\langle -\cos\theta, \sin\theta, 0\rangle \text{ m}$$

$$\mathbf{r}_B = \langle -10, -1, 0\rangle \text{ m}$$

Use these to write a formula for the force acting from A to B.

$$\mathbf{P} = P\hat{\mathbf{u}}_{\text{AB}} = 6000\frac{\mathbf{r}_B - \mathbf{r}_A}{|\mathbf{r}_B - \mathbf{r}_A|} \text{ N} = 6000\frac{\langle -10 + 8\cos\theta, -1 - 8\sin\theta, 0\rangle}{\sqrt{(-10 + 8\cos\theta)^2 + (-1 - 8\sin\theta)^2 + 0^2}} \text{ N}$$

$$= 6000\frac{\langle -10 + 8\cos\theta, -1 - 8\sin\theta, 0\rangle}{\sqrt{165 - 160\cos\theta + 16\sin\theta}} \text{ N}$$

Now calculuate the moment of this force about O.

$$\mathbf{M}_P = \mathbf{r}_B \times \mathbf{P} = \langle -10, -1, 0\rangle \times 6000\frac{\langle -10 + 8\cos\theta, -1 - 8\sin\theta, 0\rangle}{\sqrt{165 - 160\cos\theta + 16\sin\theta}} \text{ N}\cdot\text{m}$$

$$= \frac{6000}{\sqrt{165 - 160\cos\theta + 16\sin\theta}}\langle -10, -1, 0\rangle \times \langle -10 + 8\cos\theta, -1 - 8\sin\theta, 0\rangle \text{ N}\cdot\text{m}$$

$$= \frac{6000}{\sqrt{165 - 160\cos\theta + 16\sin\theta}}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -10 & -1 & 0 \\ -10 + 8\cos\theta & -1 - 8\sin\theta & 0 \end{vmatrix} \text{ N}\cdot\text{m}$$

$$= \frac{6000 \text{ N}\cdot\text{m}}{\sqrt{165 - 160\cos\theta + 16\sin\theta}}(8\cos\theta + 80\sin\theta)\hat{\mathbf{z}}$$

As a result,

$$\mathbf{M}_P = (48\,000 \text{ N} \cdot \text{m}) \frac{\cos\theta + 10\sin\theta}{\sqrt{165 - 160\cos\theta + 16\sin\theta}}\,\hat{\mathbf{z}}.$$

In order to find the value of θ that maximizes the moment, take the derivative of this function of θ and set it equal to zero.

$$\frac{d}{d\theta}\left[\frac{\cos\theta + 10\sin\theta}{\sqrt{165 - 160\cos\theta + 16\sin\theta}}\right] = 0$$

$$\frac{\left[\frac{d}{d\theta}(\cos\theta + 10\sin\theta)\right]\sqrt{165 - 160\cos\theta + 16\sin\theta} - \left(\frac{d}{d\theta}\sqrt{165 - 160\cos\theta + 16\sin\theta}\right)(\cos\theta + 10\sin\theta)}{165 - 160\cos\theta + 16\sin\theta} = 0$$

$$\frac{(-\sin\theta + 10\cos\theta)\sqrt{165 - 160\cos\theta + 16\sin\theta} - \left[\frac{1}{2}(165 - 160\cos\theta + 16\sin\theta)^{-1/2} \cdot \frac{d}{d\theta}(165 - 160\cos\theta + 16\sin\theta)\right](\cos\theta + 10\sin\theta)}{165 - 160\cos\theta + 16\sin\theta} = 0$$

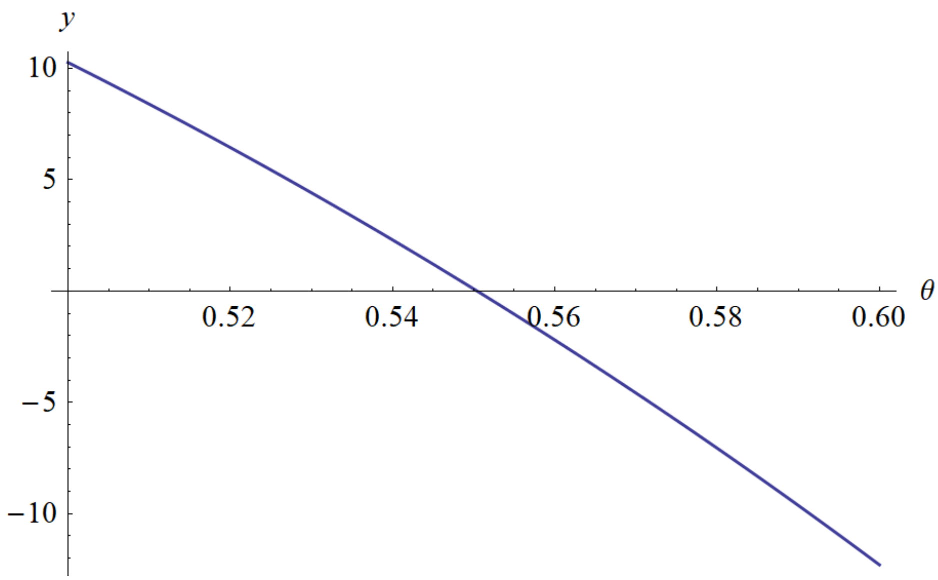
$$\frac{(-\sin\theta + 10\cos\theta)\sqrt{165 - 160\cos\theta + 16\sin\theta} - \left[\frac{1}{2}(165 - 160\cos\theta + 16\sin\theta)^{-1/2} \cdot (160\sin\theta + 16\cos\theta)\right](\cos\theta + 10\sin\theta)}{165 - 160\cos\theta + 16\sin\theta} = 0$$

$$\frac{(-\sin\theta + 10\cos\theta)(165 - 160\cos\theta + 16\sin\theta) - \left[\frac{1}{2} \cdot (160\sin\theta + 16\cos\theta)\right](\cos\theta + 10\sin\theta)}{(165 - 160\cos\theta + 16\sin\theta)^{3/2}} = 0$$

Multiply both sides by $(165 - 160\cos\theta + 16\sin\theta)^{3/2}$.

$$(-\sin\theta + 10\cos\theta)(165 - 160\cos\theta + 16\sin\theta) - (80\sin\theta + 8\cos\theta)(\cos\theta + 10\sin\theta) = 0$$

To solve for θ, plot the function on the left side versus θ and see where it crosses the horizontal axis.



$$\theta \approx 0.55 \text{ radians} \approx 31.5^\circ$$

Plugging this value of θ into the formula for $\mathbf{M}_P$ yields

$$\mathbf{M}_P = 4.80 \times 10^4 \text{ N} \cdot \text{m}$$

for the maximum moment of force $\mathbf{P}$ about O.