Problem 4-3

Given the three nonzero vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, show that if $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$, the three vectors *must* lie in the same plane.

Solution

Suppose that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = |\mathbf{A}||\mathbf{B} \times \mathbf{C}| \cos\theta = |\mathbf{A}|(|\mathbf{B}||\mathbf{C}| \sin\phi) \cos\theta = 0$$

Here θ is the angle between $\mathbf{A}$ and $\mathbf{B} \times \mathbf{C}$, and ϕ is the angle between $\mathbf{B}$ and $\mathbf{C}$. The equation above indicates that either $\sin\phi = 0$ or $\cos\theta = 0$.

If $\sin\phi = 0$, then $\mathbf{B}$ and $\mathbf{C}$ have the same direction. This direction and the direction of $\mathbf{A}$ form a plane in three-dimensional space.

If $\cos\theta = 0$, then $\mathbf{A}$ is perpendicular to the direction of $\mathbf{B} \times \mathbf{C}$. But because $\mathbf{B} \times \mathbf{C}$ itself is perpendicular to the plane of $\mathbf{B}$ and $\mathbf{C}$, $\mathbf{A}$ must lie within the plane of $\mathbf{B}$ and $\mathbf{C}$.