Problem 1.1

Use Gauss's theorem [and (1.21) if necessary] to prove the following:

(a) Any excess charge placed on a conductor must lie entirely on its surface. (A conductor by definition contains charges capable of moving freely under the action of applied electric fields.)

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it.

(c) The electric field at the surface of a conductor is normal to the surface and has a magnitude σ/ϵ_0, where σ is the charge density per unit area on the surface.

Solution

Gauss's law is

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},$$

where $\mathbf{E}$ is the electric field, ρ is the charge density, and ϵ_0 is a known constant.

Part (a)

Suppose there's an arbitrarily shaped three-dimensional solid conductor with excess charge. If any of this excess charge lies within its volume, then both sides of Gauss's law can be integrated over the volume of a Gaussian surface containing this charge.

$$\iiint_V \nabla \cdot \mathbf{E}\, dV = \iiint_V \frac{\rho}{\epsilon_0}\, dV$$

Use the divergence theorem on the left and evaluate the integral on the right.

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \sum_i q_i$$

The sum on the right is the total excess charge enclosed. Because it's nonzero, $\mathbf{E}$ is nonzero as well. A nonzero $\mathbf{E}$ makes charges move in the conductor, meaning that an electrostatic charge distribution is not possible. All of the excess charge must therefore reside on the conductor's surface, where it cannot be enclosed by an interior Gaussian surface.

Part (b)

Suppose there's an arbitrarily shaped three-dimensional hollow conductor (with inner and outer surfaces) next to a convocation of charges. The conductor experiences an applied electric field from these external charges, causing the charges free to move inside it to align on the outer surface with the applied field. This polarization produces an induced electric field that opposes the applied field. These two fields cancel one another by the principle of superposition to make $\mathbf{E} = 0$ within the conductor.

Suppose instead that there's a convocation of total charge q inside the cavity of this hollow conductor. In order to have $\mathbf{E} = 0$ inside the conductor, the free charge needs to be arranged such that $-q$ is on its inner surface. Assuming the conductor is electrically neutral, a corresponding charge $+q$ has to be on the outer surface. A Gaussian surface that encloses the entire conductor has a net charge of $q - q + q = q$, which means the conductor does not shield the exterior from charges within. If the conductor was not electrically neutral but had excess charge Q, the inner surface would have charge $-q$ and the outer surface would have charge $Q + q$ instead.

Part (c)

Faraday's law is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

For stationary charges (electrostatics), the right side is zero.

$$\nabla \times \mathbf{E} = \mathbf{0}$$

This implies the existence of a potential function $-\Phi$ that satisfies

$$\mathbf{E} = \nabla(-\Phi) = -\nabla\Phi.$$

The minus sign is arbitrary mathematically, but physically it indicates that a charge in an electric field has more potential energy upstream than it does downstream. To get the work done by the electric field on a unit charge that moves from $\mathbf{a}$ to $\mathbf{b}$, integrate both sides along a path from $\mathbf{a}$ to $\mathbf{b}$.

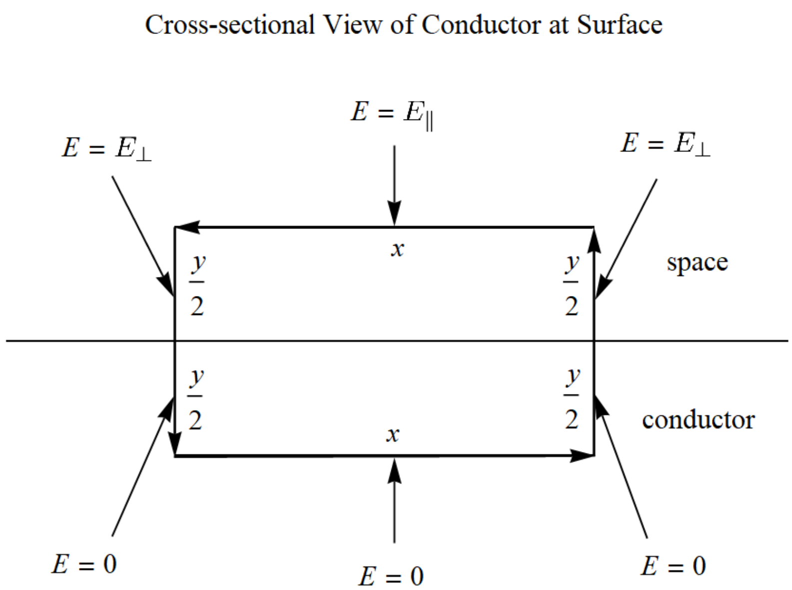
$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} &= -\int_{\mathbf{a}}^{\mathbf{b}} \nabla\Phi \cdot d\mathbf{l} \\ &= -[\Phi(\mathbf{b}) - \Phi(\mathbf{a})] \\ &= \Phi(\mathbf{a}) - \Phi(\mathbf{b})\end{aligned}$$

If the starting point is the same as the ending point, then

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1.21}$$

regardless of the path taken. This is characteristic of a conservative force.

The electric field at the surface of a conductor can be decomposed into a tangential component and a normal component: $\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}$. Consider a small rectangular loop partly inside a conductor as shown below.



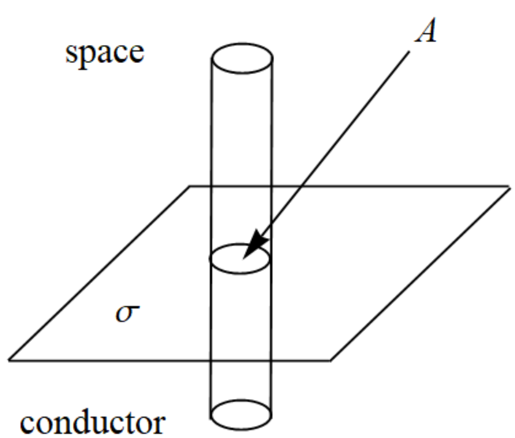
x and y are assumed to be small enough that $E_{\parallel}$ and $E_{\perp}$ do not change significantly over the lengths. Apply equation (1.21) to this loop.

$$\oint \mathbf{E} \cdot d\mathbf{l} = E_{\perp}\left(-\frac{y}{2}\right) + E_{\parallel}(-x) + E_{\perp}\left(\frac{y}{2}\right) = 0$$

Solve for the tangential component.

$$E_{\parallel} = 0$$

Therefore, the electric field only has a normal component at the surface of a conductor: $\mathbf{E} = \mathbf{E}_{\perp}$. To find it, integrate both sides of Gauss's law over the volume of a cylindrical Gaussian surface going through the surface of a conductor.



$$\iiint_V \nabla \cdot \mathbf{E}\, dV = \iiint_V \frac{\rho}{\epsilon_0}\, dV$$

Use the divergence theorem on the left side and evaluate the integral on the right.

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \frac{\sigma A}{\epsilon_0}$$

The tangential electric field is zero, so there's no electric flux through the lateral sides. $\mathbf{E} = 0$ within the conductor, so there's zero electric flux through the bottom end. On the top end, though, the electric flux is $E_\perp A$.

$$E_\perp A = \frac{\sigma A}{\epsilon_0}$$

Solve for $E_\perp$.

$$E_\perp = \frac{\sigma}{\epsilon_0}$$

Therefore, at the surface of a conductor, the magnitude of the electric field is

$$|\mathbf{E}| = \frac{\sigma}{\epsilon_0}.$$