Problem 1.10

Prove the *mean value theorem*: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered on that point.

Solution

The governing equations of the electric field are Gauss's law and Faraday's law. In the context of electrostatics with no charge distribution they are

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

This second equation implies the existence of a potential function $-\Phi$ that satisfies

$$\mathbf{E} = \nabla(-\Phi) = -\nabla\Phi.$$

The minus sign is arbitrary mathematically, but physically it indicates that a charge in an electric field has more potential energy upstream than it does downstream. Substitute this formula into Gauss's law to obtain Poisson's equation.

$$\nabla \cdot (-\nabla\Phi) = 0$$

$$-\nabla \cdot \nabla\Phi = 0$$

$$\nabla^2\Phi = 0$$

Since the right side is zero, this is actually Laplace's equation. Suppose we want to evaluate Φ at (x, y, z), a point in space. Integrate both sides over the volume of a sphere centered on this point with radius R.

$$\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\leq R^2}} \nabla^2\Phi \, dV_0 = \iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\leq R^2}} (0)\, dV_0$$

$$\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\leq R^2}} \nabla \cdot \nabla\Phi \, dV_0 = 0$$

Apply the divergence theorem to turn this volume integral into a surface integral.

$$0 = \oiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=R^2}} \nabla\Phi \cdot d\mathbf{S}_0$$

$$= \oiint\limits_{r_0=R} \nabla\Phi \cdot \hat{\mathbf{r}}_0 \, dS_0$$

$$= \oiint\limits_{r_0=R} \frac{\partial\Phi}{\partial r_0} \, dS_0$$

Note that $\hat{\mathbf{r}}_0$ is the outward radial unit vector from this sphere's center.

Write the surface integral explicitly by using spherical coordinates (r_0, ϕ_0, θ_0), where r_0 is the radial distance from the sphere's center at (x, y, z) and θ_0 is the angle from the sphere's polar axis.

$$0 = \int_0^\pi \int_0^{2\pi} \frac{\partial \Phi}{\partial r_0}\bigg|_{r_0=R} (R^2 \sin\theta_0 \, d\phi_0 \, d\theta_0)$$

Divide both sides by $4\pi R^2$.

$$0 = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\partial \Phi}{\partial r_0}\bigg|_{r_0=R} (\sin\theta_0 \, d\phi_0 \, d\theta_0)$$

This result holds for any $R > 0$, so the evaluation can be dropped. In other words, R can be thought of as a variable that is indistinguishable from r_0 on the spherical surface.

$$0 = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\partial \Phi}{\partial r_0} (\sin\theta_0 \, d\phi_0 \, d\theta_0)$$

$$\frac{d}{dr_0}\left[\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \Phi(r_0, \phi_0, \theta_0)(\sin\theta_0 \, d\phi_0 \, d\theta_0)\right] = 0$$

Integrate both sides with respect to r_0.

$$\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \Phi(r_0, \phi_0, \theta_0)(\sin\theta_0 \, d\phi_0 \, d\theta_0) = C$$

Rewrite the denominator as a double integral.

$$\frac{\int_0^\pi \int_0^{2\pi} \Phi(r_0, \phi_0, \theta_0)(\sin\theta_0 \, d\phi_0 \, d\theta_0)}{\int_0^\pi \int_0^{2\pi} (\sin\theta_0 \, d\phi_0 \, d\theta_0)} = C \tag{1}$$

This equation holds for any $r_0 > 0$, so set $r_0 = R$ and include R^2 in each integral.

$$\frac{\int_0^\pi \int_0^{2\pi} \Phi(R, \phi_0, \theta_0)(R^2 \sin\theta_0 \, d\phi_0 \, d\theta_0)}{\int_0^\pi \int_0^{2\pi} (R^2 \sin\theta_0 \, d\phi_0 \, d\theta_0)} = C \tag{2}$$

Also, take the limit of both sides of equation (1) as $r_0 \to 0$.

$$\lim_{r_0 \to 0} \frac{\int_0^\pi \int_0^{2\pi} \Phi(r_0, \phi_0, \theta_0)(\sin\theta_0 \, d\phi_0 \, d\theta_0)}{\int_0^\pi \int_0^{2\pi} (\sin\theta_0 \, d\phi_0 \, d\theta_0)} = \lim_{r_0 \to 0} C$$

$$\frac{\int_0^\pi \int_0^{2\pi} \Phi(x, y, z)(\sin\theta_0 \, d\phi_0 \, d\theta_0)}{\int_0^\pi \int_0^{2\pi} (\sin\theta_0 \, d\phi_0 \, d\theta_0)} = C$$

$\Phi(x,y,z)$ is independent of ϕ_0 and θ_0, so it can be brought in front of the integral.

$$\frac{\Phi(x,y,z)\displaystyle\int_0^{\pi}\int_0^{2\pi}(\sin\theta_0\,d\phi_0\,d\theta_0)}{\displaystyle\int_0^{\pi}\int_0^{2\pi}(\sin\theta_0\,d\phi_0\,d\theta_0)} = C$$

Cancel the common factors.

$$\Phi(x,y,z) = C \tag{3}$$

Combine equations (2) and (3).

$$\Phi(x,y,z) = \frac{\displaystyle\int_0^{\pi}\int_0^{2\pi}\Phi(R,\phi_0,\theta_0)(R^2\sin\theta_0\,d\phi_0\,d\theta_0)}{\displaystyle\int_0^{\pi}\int_0^{2\pi}(R^2\sin\theta_0\,d\phi_0\,d\theta_0)}$$

Therefore, the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.