Problem 1.11

Use Gauss's theorem to prove that at the surface of a curved charged conductor, the normal derivative of the electric field is given by

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where R_1 and R_2 are the principal radii of curvature of the surface.

Solution

Gauss's law gives the relationship between the electric field $\mathbf{E}$ and the charge density ρ.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Take the gradient of both sides.

$$\nabla(\nabla \cdot \mathbf{E}) = \nabla\left(\frac{\rho}{\epsilon_0}\right) = \frac{1}{\epsilon_0}\nabla\rho$$

Integrate both sides over the volume of an arbitrary three-dimensional conductor.

$$\iiint_V \nabla(\nabla \cdot \mathbf{E})\, dV = \frac{1}{\epsilon_0}\iiint_V \nabla\rho\, dV$$

Since $\rho = 0$ within the conductor, $\nabla\rho = \mathbf{0}$ as well, which makes the right side zero.

$$\iiint_V \nabla(\nabla \cdot \mathbf{E})\, dV = \mathbf{0}$$

Use Identity 18 on the left.

$$\oiint_S (\nabla \cdot \mathbf{E})\mathbf{n}\, dS = \mathbf{0}$$

Because the boundary of this conductor is arbitrary, the surface integral may be removed.

$$\nabla \cdot \mathbf{E} = 0 \quad \text{on } S$$

On the conductor's surface, the electric field is entirely normal: $\mathbf{E} = E\mathbf{n}$.

$$\nabla \cdot (E\mathbf{n}) = 0 \quad \text{on } S$$

Use Identity 7.

$$\mathbf{n} \cdot \nabla E + E(\nabla \cdot \mathbf{n}) = 0 \quad \text{on } S$$

$$\frac{\partial E}{\partial n} + E(\nabla \cdot \mathbf{n}) = 0 \quad \text{on } S$$

Divide both sides by E.

$$\frac{1}{E}\frac{\partial E}{\partial n} + \nabla \cdot \mathbf{n} = 0 \quad \text{on } S$$

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\nabla \cdot \mathbf{n} \quad \text{on } S$$

Write the divergence of the unit normal vector in terms of the principal radii of curvature.

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \quad \text{on } S$$