Problem 1.12

Prove *Green's reciprocation theorem*: If Φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\Phi'\, d^3x + \int_S \sigma\Phi'\, da = \int_V \rho'\Phi\, d^3x + \int_S \sigma'\Phi\, da$$

Solution

The governing equations of the electric field are Gauss's law and Faraday's law. In the context of electrostatics they are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

This second equation implies the existence of a potential function $-\Phi$ that satisfies

$$\mathbf{E} = \nabla(-\Phi) = -\nabla\Phi.$$

The minus sign is arbitrary mathematically, but physically it indicates that a charge in an electric field has more potential energy upstream than it does downstream. Substitute this formula into Gauss's law to obtain Poisson's equation.

$$\nabla \cdot (-\nabla\Phi) = \frac{\rho}{\epsilon_0}$$

$$-\nabla \cdot \nabla\Phi = \frac{\rho}{\epsilon_0}$$

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

The boundary condition at a surface with charge density σ is

$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \mathbf{n} = \frac{\sigma}{\epsilon_0},$$

where $\mathbf{E}_2$ and $\mathbf{E}_1$ are the electric fields right outside and right inside the volume, respectively. Write it in terms of potential.

$$[(-\nabla\Phi_2) - (-\nabla\Phi_1)] \cdot \mathbf{n} = \frac{\sigma}{\epsilon_0}$$

$$(\nabla\Phi_1 - \nabla\Phi_2) \cdot \mathbf{n} = \frac{\sigma}{\epsilon_0}$$

$$\frac{\partial\Phi_1}{\partial n} - \frac{\partial\Phi_2}{\partial n} = \frac{\sigma}{\epsilon_0}$$

$$\frac{\partial}{\partial n}(\Phi_1 - \Phi_2) = \frac{\sigma}{\epsilon_0}$$

Since Φ represents the potential within V, we set $\Phi = \Phi_1 - \Phi_2$. (If it were the potential outside V, we would set $\Phi = \Phi_2 - \Phi_1$.) Therefore, the given boundary value problems are as follows.

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \quad \text{in } V \qquad\qquad \nabla^2\Phi' = -\frac{\rho'}{\epsilon_0} \quad \text{in } V$$

$$\frac{\partial\Phi}{\partial n} = \frac{\sigma}{\epsilon_0} \quad \text{on } S \qquad\qquad \frac{\partial\Phi'}{\partial n} = \frac{\sigma'}{\epsilon_0} \quad \text{on } S$$

For any two functions, u and v, that are defined within a volume V and on its bounding surface S, Green's second identity states that

$$\iiint_V \left(u\nabla^2 v - v\nabla^2 u\right) dV = \oiint_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS.$$

Let $u = \Phi$ and $v = \Phi'$.

$$\iiint_V \left(\Phi\nabla^2 \Phi' - \Phi'\nabla^2 \Phi\right) dV = \oiint_S \left(\Phi\frac{\partial \Phi'}{\partial n} - \Phi'\frac{\partial \Phi}{\partial n}\right) dS$$

Substitute the formulas for the derivatives.

$$\iiint_V \left[\Phi\left(-\frac{\rho'}{\epsilon_0}\right) - \Phi'\left(-\frac{\rho}{\epsilon_0}\right)\right] dV = \oiint_S \left[\Phi\left(\frac{\sigma'}{\epsilon_0}\right) - \Phi'\left(\frac{\sigma}{\epsilon_0}\right)\right] dS$$

Split up the integrals.

$$-\frac{1}{\epsilon_0}\iiint_V \rho'\Phi\, dV + \frac{1}{\epsilon_0}\iiint_V \rho\Phi'\, dV = \frac{1}{\epsilon_0}\oiint_S \sigma'\Phi\, dS - \frac{1}{\epsilon_0}\oiint_S \sigma\Phi'\, dS$$

Bring the terms with Φ' to the left and bring the terms with Φ to the right.

$$\frac{1}{\epsilon_0}\iiint_V \rho\Phi'\, dV + \frac{1}{\epsilon_0}\oiint_S \sigma\Phi'\, dS = \frac{1}{\epsilon_0}\iiint_V \rho'\Phi\, dV + \frac{1}{\epsilon_0}\oiint_S \sigma'\Phi\, dS$$

Therefore, multiplying both sides by ϵ_0,

$$\iiint_V \rho\Phi'\, dV + \oiint_S \sigma\Phi'\, dS = \iiint_V \rho'\Phi\, dV + \oiint_S \sigma'\Phi\, dS.$$