Problem 1.2

The Dirac delta function in three dimensions can be taken as the improper limit as $\alpha \to 0$ of the Gaussian function

$$D(\alpha; x, y, z) = (2\pi)^{-3/2}\alpha^{-3} \exp\left[-\frac{1}{2\alpha^2}(x^2 + y^2 + z^2)\right]$$

Consider a general orthogonal coordinate system specified by the surfaces u = constant, v = constant, w = constant, with length elements du/U, dv/V, dw/W in the three perpendicular directions. Show that

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(u - u')\delta(v - v')\delta(w - w') \cdot UVW$$

by considering the limit of the Gaussian above. Note that as $\alpha \to 0$ only the infinitesimal length element need be used for the distance between the points in the exponent.

Solution

Observe that $D(\alpha; x, y, z)$ can be written as a product of three one-dimensional functions.

$$\begin{aligned} D(\alpha; x, y, z) &= (2\pi)^{-3/2}\alpha^{-3} \exp\left[-\frac{1}{2\alpha^2}(x^2 + y^2 + z^2)\right] \\ &= \frac{1}{(2\pi)^{3/2}}\frac{1}{\alpha^3}\exp\left(-\frac{x^2}{2\alpha^2} - \frac{y^2}{2\alpha^2} - \frac{z^2}{2\alpha^2}\right) \\ &= \left[\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left(-\frac{x^2}{2\alpha^2}\right)\right]\left[\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left(-\frac{y^2}{2\alpha^2}\right)\right]\left[\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left(-\frac{z^2}{2\alpha^2}\right)\right] \\ &= D_1(\alpha; x)D_1(\alpha; y)D_1(\alpha; z) \end{aligned}$$

By definition, the delta function satisfies the following two properties.

$$\begin{aligned} &(1) \qquad \delta(x - x') = 0, \qquad x \neq x' \\ &(2) \qquad \int_{-\infty}^{\infty} \delta(x - x')\, dx = 1 \end{aligned}$$

The aim, then, is to show that $D_1(\alpha; x)$ satisfies them in the limit as $\alpha \to 0$ (more accurately, as $\alpha \to 0^+$). Start with the first one, assuming $x \neq x'$. Note that $\alpha^2 \ln \alpha \to 0$ as $\alpha \to 0^+$.

$$\begin{aligned} \lim_{\alpha \to 0^+} D_1(\alpha; x - x') = \lim_{\alpha \to 0^+} \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left[-\frac{(x - x')^2}{2\alpha^2}\right] &= \frac{1}{\sqrt{2\pi}}\lim_{\alpha \to 0^+}\exp\left[\ln\left(\frac{1}{\alpha}\right)\right]\exp\left[-\frac{(x - x')^2}{2\alpha^2}\right] \\ &= \frac{1}{\sqrt{2\pi}}\lim_{\alpha \to 0^+}\exp\left[\ln\left(\frac{1}{\alpha}\right) - \frac{(x - x')^2}{2\alpha^2}\right] \\ &= \frac{1}{\sqrt{2\pi}}\lim_{\alpha \to 0^+}\exp\left[-\frac{2\alpha^2 \ln \alpha + (x - x')^2}{2\alpha^2}\right] \\ &= \frac{1}{\sqrt{2\pi}}\lim_{\alpha \to 0^+}\exp\left[-\frac{(x - x')^2}{2\alpha^2}\right] = \frac{1}{\sqrt{2\pi}}e^{-\infty} = 0 \end{aligned}$$

Now check the second property.

$$\lim_{\alpha\to 0^+}\int_{-\infty}^{\infty} D_1(\alpha; x-x')\,dx = \lim_{\alpha\to 0^+}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left[-\frac{(x-x')^2}{2\alpha^2}\right]dx = \frac{1}{\sqrt{2\pi}}\lim_{\alpha\to 0^+}\frac{1}{\alpha}\int_{-\infty}^{\infty}\exp\left(-\frac{q^2}{2\alpha^2}\right)dq$$

$$= \frac{2}{\sqrt{2\pi}}\lim_{\alpha\to 0^+}\frac{1}{\alpha}\int_{0}^{\infty}\exp\left[-\frac{q^2}{(\sqrt{2}\alpha)^2}\right]dq$$

$$= \frac{2}{\sqrt{2\pi}}\lim_{\alpha\to 0^+}\frac{1}{\alpha}\cdot\sqrt{\pi}\left(\frac{\sqrt{2}\alpha}{2}\right)$$

$$= 1$$

Consequently,

$$\lim_{\alpha\to 0^+} D_1(\alpha; x-x') = \delta(x-x')$$

and

$$\lim_{\alpha\to 0^+} D_1(\alpha; y-y') = \delta(y-y')$$

and

$$\lim_{\alpha\to 0^+} D_1(\alpha; z-z') = \delta(z-z').$$

By taking the limit of $D(\alpha; x-x', y-y', z-z')$ as $\alpha\to 0^+$, we obtain the delta function in three dimensions.

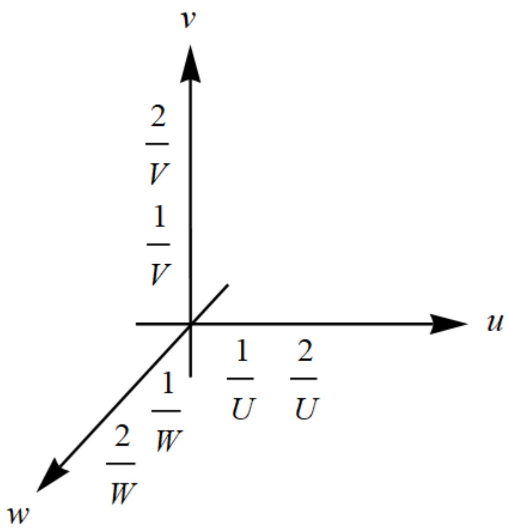
$$\lim_{\alpha\to 0^+} D(\alpha; x-x', y-y', z-z') = \lim_{\alpha\to 0^+} D_1(\alpha; x-x')D_1(\alpha; y-y')D_1(\alpha; z-z')$$

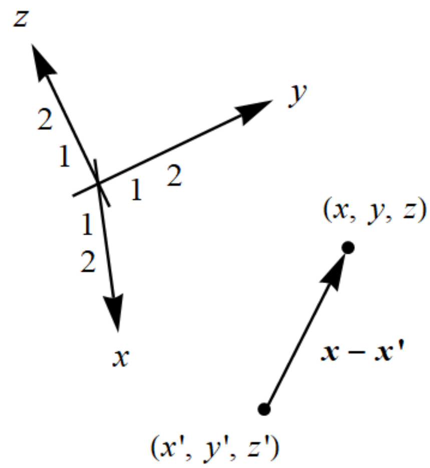
$$= \left[\lim_{\alpha\to 0^+} D_1(\alpha; x-x')\right]\left[\lim_{\alpha\to 0^+} D_1(\alpha; y-y')\right]\left[\lim_{\alpha\to 0^+} D_1(\alpha; z-z')\right]$$

$$= \delta(x-x')\delta(y-y')\delta(z-z')$$

$$= \delta(\mathbf{x}-\mathbf{x}')$$

The coordinate system described by $u =$ constant, $v =$ constant, and $w =$ constant is formed from the intersection of three planes, similar to the Cartesian coordinate system.





U, V, and W are scale factors in the u, v, and w directions, respectively. In the xyz-coordinate system,

$$\mathbf{x} - \mathbf{x}' = (x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}.$$

Suppose we want to use the u-axis as a ruler to measure the $x - x'$ distance. The measured length, $u - u'$, will be $x - x'$ multiplied by U.

$$u - u' = U(x - x')$$

Similarly, measuring $y - y_0$ with the v-axis and measuring $z - z_0$ with the w-axis results in

$$v - v' = V(y - y')$$

$$w - w' = W(z - z').$$

Now consider the integral of $D(\alpha; x - x', y - y', z - z')$ over all of space in the limit as $\alpha \to 0^+$.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\mathbf{x} - \mathbf{x}')\,dx\,dy\,dz \\
&= \lim_{\alpha\to 0^+}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(\alpha; x - x', y - y', z - z')\,dx\,dy\,dz \\
&= \lim_{\alpha\to 0^+}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left[-\frac{(x - x')^2}{2\alpha^2}\right]\right\}\left\{\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left[-\frac{(y - y')^2}{2\alpha^2}\right]\right\} \\
&\qquad \times \left\{\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left[-\frac{(z - z')^2}{2\alpha^2}\right]\right\}dx\,dy\,dz \\
&= \lim_{\alpha\to 0^+}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{u - u'}{U}\right)^2\right]\right\}\left\{\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{v - v'}{V}\right)^2\right]\right\} \\
&\qquad \times \left\{\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha}\exp\left[-\frac{1}{2\alpha^2}\left(\frac{w - w'}{W}\right)^2\right]\right\}\left(\frac{du}{U}\right)\left(\frac{dv}{V}\right)\left(\frac{dw}{W}\right)
\end{aligned}$$

Make the additional change of variables, $k = u - u'$ and $l = v - v'$ and $m = w - w'$.

$$\begin{aligned}
1 &= \lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \frac{1}{\alpha} \exp\left(-\frac{k^2}{2\alpha^2 U^2} \right) \right] \left[\frac{1}{\sqrt{2\pi}} \frac{1}{\alpha} \exp\left(-\frac{l^2}{2\alpha^2 V^2} \right) \right] \\
&\qquad \times \left[\frac{1}{\sqrt{2\pi}} \frac{1}{\alpha} \exp\left(-\frac{m^2}{2\alpha^2 W^2} \right) \right] \left(\frac{dk}{U} \right) \left(\frac{dl}{V} \right) \left(\frac{dm}{W} \right) \\
&= \lim_{\alpha \to 0^+} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \left[\frac{2}{\sqrt{2\pi}} \frac{1}{\alpha} \exp\left(-\frac{k^2}{2\alpha^2 U^2} \right) \right] \left[\frac{2}{\sqrt{2\pi}} \frac{1}{\alpha} \exp\left(-\frac{l^2}{2\alpha^2 V^2} \right) \right] \\
&\qquad \times \left[\frac{2}{\sqrt{2\pi}} \frac{1}{\alpha} \exp\left(-\frac{m^2}{2\alpha^2 W^2} \right) \right] \left(\frac{dk}{U} \right) \left(\frac{dl}{V} \right) \left(\frac{dm}{W} \right) \\
&= \lim_{\alpha \to 0^+} \left[\frac{2}{\sqrt{2\pi}} \frac{1}{\alpha} \sqrt{\pi} \cdot \left(\frac{\sqrt{2}\alpha U}{2} \right) \right] \left[\frac{2}{\sqrt{2\pi}} \frac{1}{\alpha} \sqrt{\pi} \cdot \left(\frac{\sqrt{2}\alpha V}{2} \right) \right] \\
&\qquad \times \left[\frac{2}{\sqrt{2\pi}} \frac{1}{\alpha} \sqrt{\pi} \cdot \left(\frac{\sqrt{2}\alpha W}{2} \right) \right] \left(\frac{1}{U} \right) \left(\frac{1}{V} \right) \left(\frac{1}{W} \right) \\
&= \lim_{\alpha \to 0^+} \left[\frac{2}{\sqrt{2\pi}} \frac{1}{\alpha} \sqrt{\pi} \cdot \left(\frac{\sqrt{2}\alpha}{2} \right) \right] \left[\frac{2}{\sqrt{2\pi}} \frac{1}{\alpha} \sqrt{\pi} \cdot \left(\frac{\sqrt{2}\alpha}{2} \right) \right] \\
&\qquad \times \left[\frac{2}{\sqrt{2\pi}} \frac{1}{\alpha} \sqrt{\pi} \cdot \left(\frac{\sqrt{2}\alpha}{2} \right) \right] \cdot UVW \left(\frac{1}{U} \right) \left(\frac{1}{V} \right) \left(\frac{1}{W} \right) \\
&= \lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha} \exp\left[-\frac{(u-u')^2}{2\alpha^2} \right] \right\} \left\{ \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha} \exp\left[-\frac{(v-v')^2}{2\alpha^2} \right] \right\} \\
&\qquad \times \left\{ \frac{1}{\sqrt{2\pi}} \frac{1}{\alpha} \exp\left[-\frac{(w-w')^2}{2\alpha^2} \right] \right\} \cdot UVW \left(\frac{du}{U} \right) \left(\frac{dv}{V} \right) \left(\frac{dw}{W} \right) \\
&= \lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(\alpha; u-u', v-v', w-w') \cdot UVW \left(\frac{du}{U} \right) \left(\frac{dv}{V} \right) \left(\frac{dw}{W} \right) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(u-u')\delta(v-v')\delta(w-w') \cdot UVW \, dx \, dy \, dz
\end{aligned}$$

Therefore,

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(u-u')\delta(v-v')\delta(w-w') \cdot UVW.$$