Problem 1.3

Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

(a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.

(b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b.

(c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R.

(d) The same as part (c), but using spherical coordinates.

Solution

Part (a)

Since the spherical shell exists entirely at $r = R$, only the delta function $\delta(r - R)$ is necessary to describe the charge density.

$$\rho(\mathbf{x}) = A\delta(r - R)$$

A is a normalization constant: Determine it by requiring the integral over a volume containing the shell to be Q.

$$\begin{aligned} Q = \iiint \rho(\mathbf{x})\, d\mathbf{x} &= \int_0^\pi \int_0^{2\pi} \int_0^\infty A\delta(r - R)(r^2 \sin\theta\, dr\, d\phi\, d\theta) \\ &= A\left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)\left[\int_0^\infty \delta(r - R) r^2\, dr\right] \\ &= A(2)(2\pi)(R^2) \end{aligned}$$

Therefore,

$$A = \frac{Q}{4\pi R^2},$$

and the charge density is

$$\rho(\mathbf{x}) = \frac{Q}{4\pi R^2}\delta(r - R).$$

Part (b)

Since the cylindrical shell exists entirely at $r = b$, only the delta function $\delta(r - b)$ is necessary to describe the charge density.

$$\rho(\mathbf{x}) = A\delta(r - b)$$

A is a normalization constant: Determine it by requiring the integral over a volume containing one unit of the shell's length to be λ.

$$\begin{aligned}
\lambda = \iiint \rho(\mathbf{x})\, d\mathbf{x} &= \int_0^1 \int_0^{2\pi} \int_0^\infty A\delta(r - b)(r\, dr\, d\phi\, dz) \\
&= A\left(\int_0^1 dz\right)\left(\int_0^{2\pi} d\phi\right)\left[\int_0^\infty \delta(r - b) r\, dr\right] \\
&= A(1)(2\pi)(b)
\end{aligned}$$

Therefore,

$$A = \frac{\lambda}{2\pi b},$$

and the charge density is

$$\rho(\mathbf{x}) = \frac{\lambda}{2\pi b}\delta(r - b).$$

Part (c)

Assume the circular disc is centered at the origin in the xy-plane with the z axis perpendicular to it. Since it exists entirely at $z = 0$, a delta function $\delta(z)$ is needed in the charge density. The disc exists over an interval of r, from 0 to R, so the Heaviside function is also needed.

$$\rho(\mathbf{x}) = A\left[H(r) - H(r - R)\right]\delta(z)$$

$+H(r)$ makes the density nonzero at $r = 0$ and beyond, and $-H(r - R)$ makes it zero again at $r = R$ and beyond. Alternatively, $H(R - r)$ can be used for $H(r) - H(r - R)$. A is a normalization constant: Determine it by requiring the integral over a volume containing the disc to be Q.

$$\begin{aligned}
Q = \iiint \rho(\mathbf{x})\, d\mathbf{x} &= \int_{-1}^1 \int_0^{2\pi} \int_0^\infty A\left[H(r) - H(r - R)\right]\delta(z)(r\, dr\, d\phi\, dz) \\
&= A\left[\int_{-1}^1 \delta(z)\, dz\right]\left(\int_0^{2\pi} d\phi\right)\left\{\int_0^\infty \left[H(r) - H(r - R)\right] r\, dr\right\} \\
&= A(1)(2\pi)\left[\int_0^R (1) r\, dr + \int_R^\infty (0) r\, dr\right] \\
&= A(1)(2\pi)\left(\frac{R^2}{2}\right)
\end{aligned}$$

Therefore,

$$A = \frac{Q}{\pi R^2},$$

and the charge density is

$$\rho(\mathbf{x}) = \frac{Q}{\pi R^2}\left[H(r) - H(r - R)\right]\delta(z).$$

Part (d)

Assume the circular disc is centered at the origin in the xy-plane with the z axis perpendicular to it. Since it exists entirely at the polar angle $\theta = \pi/2$, a delta function $\delta(\theta - \pi/2)$ is needed in the charge density. The disc exists over an interval of r, from 0 to R, so the Heaviside function is also needed.

$$\rho(\mathbf{x}) = A\,[H(r) - H(r-R)]\,\delta\left(\theta - \frac{\pi}{2}\right)$$

$+H(r)$ makes the density nonzero at $r = 0$ and beyond, and $-H(r-R)$ makes it zero again at $r = R$ and beyond. Alternatively, $H(R-r)$ can be used for $H(r) - H(r-R)$. A is a normalization constant: Determine it by requiring the integral over a volume containing the disc to be Q.

$$\begin{aligned} Q = \iiint \rho(\mathbf{x})\,d\mathbf{x} &= \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} A\,[H(r) - H(r-R)]\,\delta\left(\theta - \frac{\pi}{2}\right)(r^2 \sin\theta\,dr\,d\phi\,d\theta) \\ &= A\left[\int_0^{\pi} \delta\left(\theta - \frac{\pi}{2}\right)\sin\theta\,d\theta\right]\left(\int_0^{2\pi} d\phi\right)\left\{\int_0^{\infty} [H(r) - H(r-R)]\,r^2\,dr\right\} \\ &= A\left(\sin\frac{\pi}{2}\right)(2\pi)\left[\int_0^{R}(1)r^2\,dr + \int_R^{\infty}(0)r^2\,dr\right] \\ &= A(1)(2\pi)\left(\frac{R^3}{3}\right) \end{aligned}$$

Therefore,

$$A = \frac{3Q}{2\pi R^3},$$

and the charge density is

$$\rho(\mathbf{x}) = \frac{3Q}{2\pi R^3}\,[H(r) - H(r-R)]\,\delta\left(\theta - \frac{\pi}{2}\right).$$