Problem 1.4

Each of three charged spheres of radius a, one conducting, one having a uniform charge density within its volume, and one having a spherically symmetric charge density that varies radially as r^n $(n > -3)$, has a total charge Q. Use Gauss's theorem to obtain the electric fields both inside and outside each sphere. Sketch the behavior of the fields as a function of radius for the first two spheres, and for the third with $n = -2, +2$.

Solution

Gauss's law gives the relationship between the electric field $\mathbf{E}$ and the charge density ρ.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

The Conducting Sphere

All of the excess charge in this sphere lies on the surface because the sphere is conducting (Problem 1.1), so the charge density is (Problem 1.3)

$$\rho(r) = \frac{Q}{4\pi a^2}\delta(r-a).$$

Substitute this formula into Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{Q}{4\pi\epsilon_0 a^2}\delta(r-a)$$

Integrate both sides over the volume of a concentric sphere with radius r.

$$\iiint\limits_{\substack{x'^2+y'^2\\+z'^2\leq r^2}} \nabla \cdot \mathbf{E}\, dV' = \int_0^\pi \int_0^{2\pi} \int_0^r \frac{Q}{4\pi\epsilon_0 a^2}\delta(r'-a)(r'^2 \sin\theta'\, dr'\, d\phi'\, d\theta')$$

Apply the divergence theorem on the left and then simplify both sides.

$$\oiint\limits_{\substack{x'^2+y'^2\\+z'^2=r^2}} \mathbf{E} \cdot d\mathbf{S}' = \frac{Q}{4\pi\epsilon_0 a^2}\left(\int_0^\pi \sin\theta'\, d\theta'\right)\left(\int_0^{2\pi} d\phi'\right)\left[\int_0^r \delta(r'-a)r'^2\, dr'\right]$$

$$\oiint\limits_{r'=r} \mathbf{E} \cdot \hat{\mathbf{r}}\, dS' = \frac{Q}{4\pi\epsilon_0 a^2}(2)(2\pi)\int_0^r \delta(r'-a)r'^2\, dr'$$

$$\oiint\limits_{r'=r} E_r\, dS' = \frac{Q}{\epsilon_0 a^2}\int_0^r \delta(r'-a)r'^2\, dr'$$

$$E_r \oiint\limits_{r'=r} dS' = \begin{cases} 0 & \text{if } r < a \\ \frac{Q}{\epsilon_0 a^2}(a^2) & \text{if } r > a \end{cases}$$

$$E_r(4\pi r^2) = \frac{Q}{\epsilon_0}H(r-a)$$

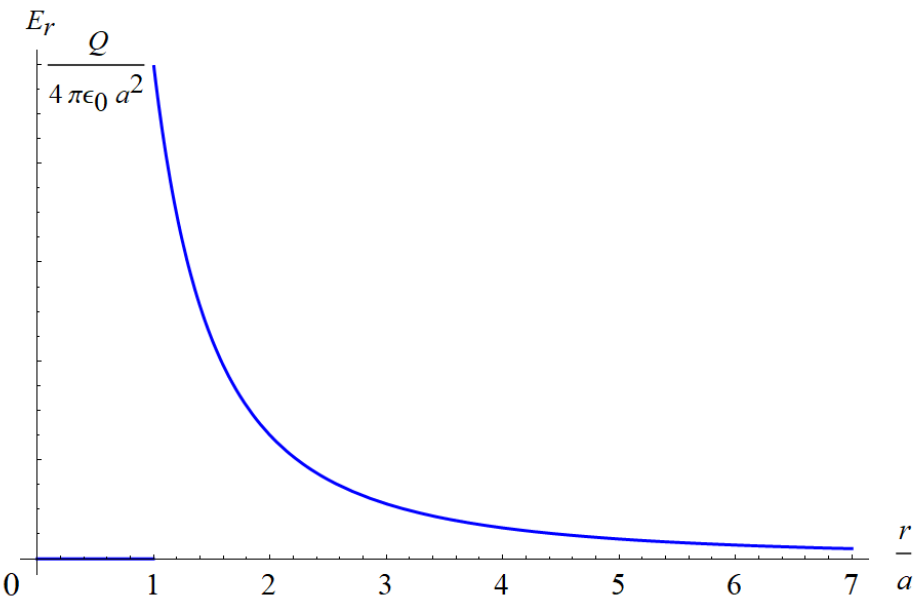
Solve for E_r, the radial component of the electric field.

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2} H(r-a) = \frac{Q}{4\pi\epsilon_0 a^2 \left(\frac{r}{a}\right)^2} H\left(\frac{r}{a} - 1\right)$$

Therefore, the electric field around a conducting sphere with radius a and charge Q is

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} H(r-a)\hat{\mathbf{r}}.}$$

Below is a plot of E_r versus r/a to illustrate the field's behavior.



The field is zero inside the sphere and falls off as $1/r^2$ outside of it.

The Constant Density Sphere

For a nonconducting sphere that has charge uniformly distributed within it, the charge density is

$$\rho(r) = \frac{Q}{\frac{4}{3}\pi a^3} H(a-r) = \frac{3Q}{4\pi a^3} H(a-r).$$

Substitute this formula into Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{3Q}{4\pi\epsilon_0 a^3} H(a-r)$$

Integrate both sides over the volume of a concentric sphere with radius r.

$$\iiint\limits_{\substack{x'^2+y'^2\\+z'^2\leq r^2}} \nabla \cdot \mathbf{E}\, dV' = \int_0^\pi \int_0^{2\pi} \int_0^r \frac{3Q}{4\pi\epsilon_0 a^3} H(a-r')(r'^2 \sin\theta'\, dr'\, d\phi'\, d\theta')$$

Apply the divergence theorem on the left and then simplify both sides.

$$\oiint\limits_{\substack{x'^2+y'^2\\+z'^2=r^2}} \mathbf{E} \cdot d\mathbf{S}' = \frac{3Q}{4\pi\epsilon_0 a^3} \left(\int_0^\pi \sin\theta'\, d\theta'\right) \left(\int_0^{2\pi} d\phi'\right) \left[\int_0^r H(a-r') r'^2\, dr'\right]$$

$$\oiint\limits_{r'=r} \mathbf{E} \cdot \hat{\mathbf{r}}\, dS' = \begin{cases} \dfrac{3Q}{4\pi\epsilon_0 a^3}(2)(2\pi) \displaystyle\int_0^r (1) r'^2\, dr' & \text{if } r < a \\ \dfrac{3Q}{4\pi\epsilon_0 a^3}(2)(2\pi) \left[\displaystyle\int_0^a (1) r'^2\, dr' + \int_a^r (0) r'^2\, dr'\right] & \text{if } r > a \end{cases}$$

$$\oiint\limits_{r'=r} E_r\, dS' = \begin{cases} \dfrac{3Q}{\epsilon_0 a^3}\left(\dfrac{r^3}{3}\right) & \text{if } r < a \\ \dfrac{3Q}{\epsilon_0 a^3}\left(\dfrac{a^3}{3}\right) & \text{if } r > a \end{cases}$$

$$E_r \oiint\limits_{r'=r} dS' = \begin{cases} \dfrac{Q}{\epsilon_0}\left(\dfrac{r}{a}\right)^3 & \text{if } r < a \\ \dfrac{Q}{\epsilon_0} & \text{if } r > a \end{cases}$$

$$E_r(4\pi r^2) = \frac{Q}{\epsilon_0}\left\{\left(\frac{r}{a}\right)^3 [H(r) - H(r-a)] + 1H(r-a)\right\}$$

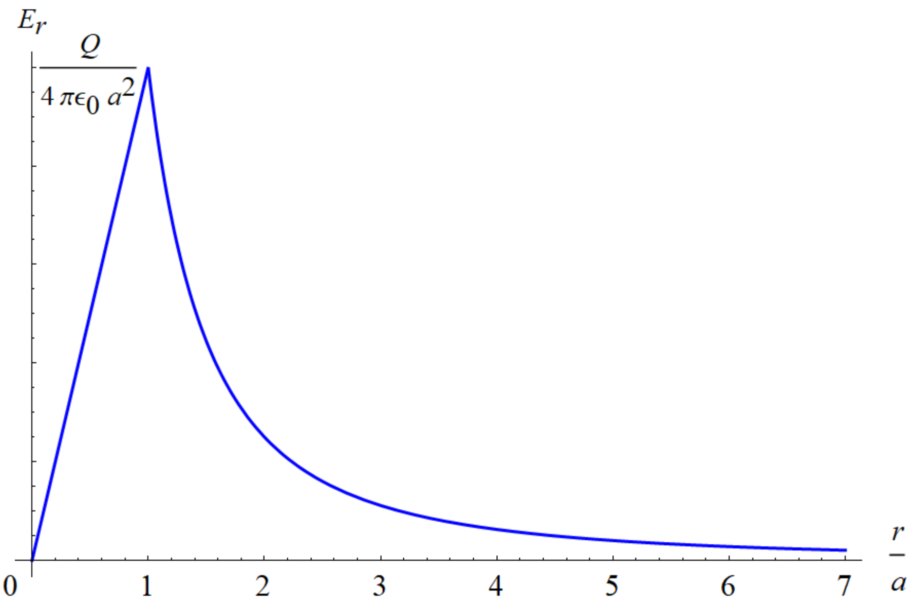
$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}\left\{\left(\frac{r}{a}\right)^3 H(r) + \left[1 - \left(\frac{r}{a}\right)^3\right] H(r-a)\right\}$$

$$= \frac{Q}{4\pi\epsilon_0 a^2 \left(\frac{r}{a}\right)^2}\left\{\left(\frac{r}{a}\right)^3 H\left(\frac{r}{a}\right) + \left[1 - \left(\frac{r}{a}\right)^3\right] H\left(\frac{r}{a} - 1\right)\right\}$$

Therefore, the electric field around a nonconducting sphere of radius a with uniformly distributed charge Q is

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\left\{\left(\frac{r}{a}\right)^3 H(r) + \left[1 - \left(\frac{r}{a}\right)^3\right] H(r-a)\right\}\hat{\mathbf{r}}.}$$

Below is a plot of E_r versus r/a to illustrate the field's behavior.



The field increases linearly inside the sphere and falls off as $1/r^2$ outside of it.

The Variable Density Sphere

Suppose a nonconducting sphere has charge radially distributed within it according to a power law.

$$\rho(r) = Ar^n H(a-r), \quad n > -3$$

Normalize this density to determine A.

$$\begin{aligned} Q = \iiint \rho\, dV &= \int_0^\pi \int_0^{2\pi} \int_0^a Ar^n H(a-r)(r^2 \sin\theta\, dr\, d\phi\, d\theta) \\ &= A\left(\int_0^\pi \sin\theta\, d\theta\right)\left(\int_0^{2\pi} d\phi\right)\left(\int_0^a r^{n+2}\, dr\right) \\ &= A(2)(2\pi)\left(\frac{a^{n+3}}{n+3}\right) \end{aligned}$$

Consequently,

$$A = \frac{(n+3)Q}{4\pi a^{n+3}},$$

and the charge density is

$$\rho(r) = \frac{(n+3)Q}{4\pi a^{n+3}} r^n H(a-r).$$

Substitute this formula into Gauss's law.

$$\nabla \cdot \mathbf{E} = \frac{(n+3)Q}{4\pi\epsilon_0 a^{n+3}} r^n H(a-r)$$

Integrate both sides over the volume of a concentric sphere with radius r.

$$\iiint\limits_{\substack{x'^2+y'^2 \\ +z'^2 \le r^2}} \nabla \cdot \mathbf{E}\, dV' = \int_0^\pi \int_0^{2\pi} \int_0^r \frac{(n+3)Q}{4\pi\epsilon_0 a^{n+3}} r'^n H(a-r')(r'^2 \sin\theta'\, dr'\, d\phi'\, d\theta')$$

$$\oiint\limits_{\substack{x'^2+y'^2 \\ +z'^2 = r^2}} \mathbf{E} \cdot d\mathbf{S}' = \frac{(n+3)Q}{4\pi\epsilon_0 a^{n+3}} \left(\int_0^\pi \sin\theta'\, d\theta'\right)\left(\int_0^{2\pi} d\phi'\right)\left[\int_0^r H(a-r') r'^{n+2}\, dr'\right]$$

$$\oiint\limits_{r'=r} \mathbf{E} \cdot \hat{\mathbf{r}}\, dS' = \begin{cases} \dfrac{(n+3)Q}{4\pi\epsilon_0 a^{n+3}}(2)(2\pi)\displaystyle\int_0^r (1) r'^{n+2}\, dr' & \text{if } r < a \\ \dfrac{(n+3)Q}{4\pi\epsilon_0 a^{n+3}}(2)(2\pi)\left[\displaystyle\int_0^a (1) r'^{n+2}\, dr' + \int_a^r (0) r'^{n+2}\, dr'\right] & \text{if } r > a \end{cases}$$

$$\oiint\limits_{r'=r} E_r\, dS' = \begin{cases} \dfrac{(n+3)Q}{\epsilon_0 a^{n+3}}\left(\dfrac{r^{n+3}}{n+3}\right) & \text{if } r < a \\ \dfrac{(n+3)Q}{\epsilon_0 a^{n+3}}\left(\dfrac{a^{n+3}}{n+3}\right) & \text{if } r > a \end{cases}$$

Because of the spherical symmetry, the electric field has the same magnitude everywhere on the surface and can be pulled in front of the integral.

$$E_r \oiint_{r'=r} dS' = \begin{cases} \dfrac{Q}{\epsilon_0}\left(\dfrac{r}{a}\right)^{n+3} & \text{if } r < a \\ \\ \dfrac{Q}{\epsilon_0} & \text{if } r > a \end{cases}$$

$$E_r(4\pi r^2) = \frac{Q}{\epsilon_0}\left\{\left(\frac{r}{a}\right)^{n+3}[H(r) - H(r-a)] + 1H(r-a)\right\}$$

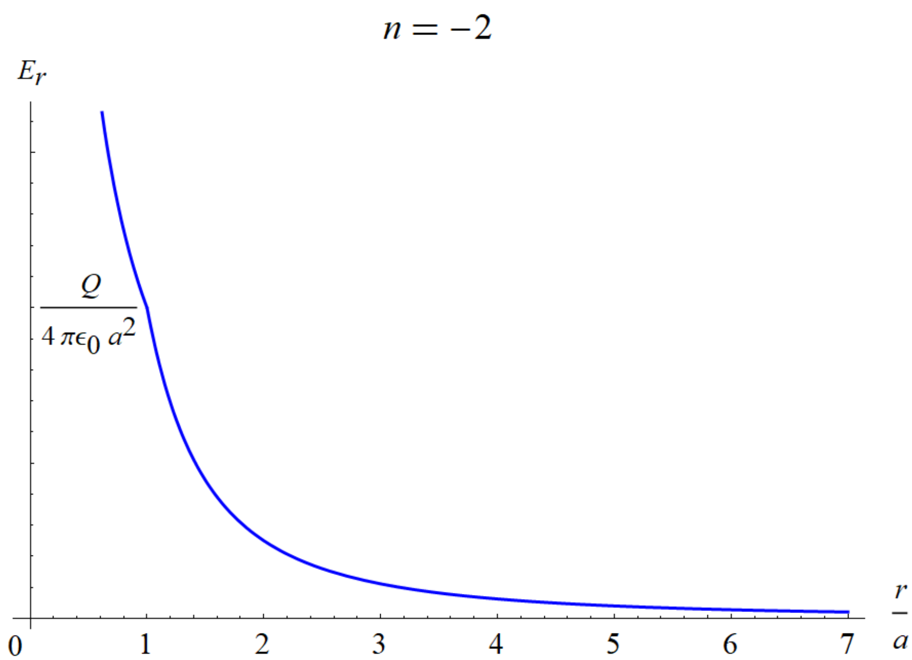
$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}\left\{\left(\frac{r}{a}\right)^{n+3} H(r) + \left[1 - \left(\frac{r}{a}\right)^{n+3}\right] H(r-a)\right\}$$

$$= \frac{Q}{4\pi\epsilon_0 a^2\left(\frac{r}{a}\right)^2}\left\{\left(\frac{r}{a}\right)^{n+3} H\left(\frac{r}{a}\right) + \left[1 - \left(\frac{r}{a}\right)^{n+3}\right] H\left(\frac{r}{a} - 1\right)\right\}$$

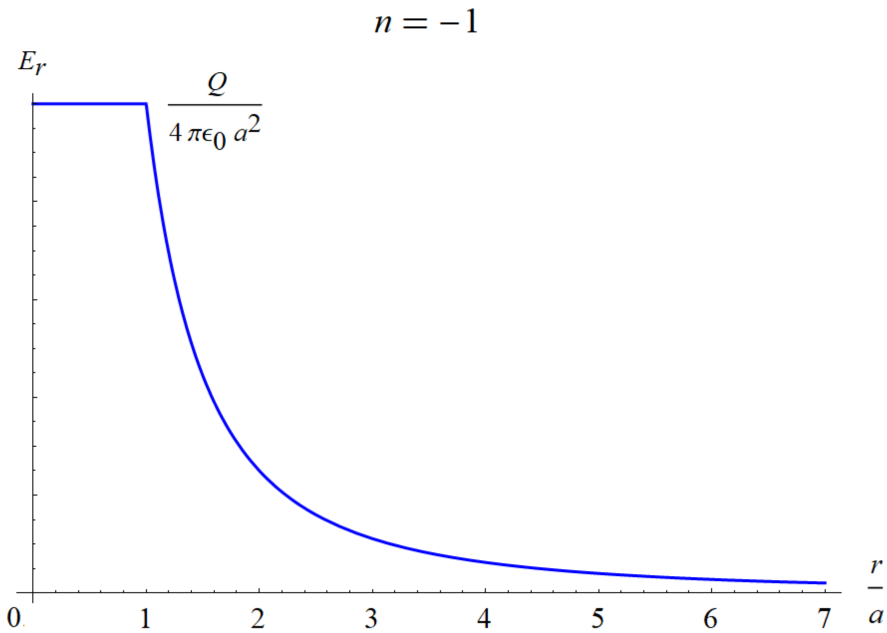
Therefore, the electric field around a nonconducting sphere of radius a with charge Q distributed radially according to a power law is

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\left\{\left(\frac{r}{a}\right)^{n+3} H(r) + \left[1 - \left(\frac{r}{a}\right)^{n+3}\right] H(r-a)\right\}\hat{\mathbf{r}}, \quad n > -3.}$$

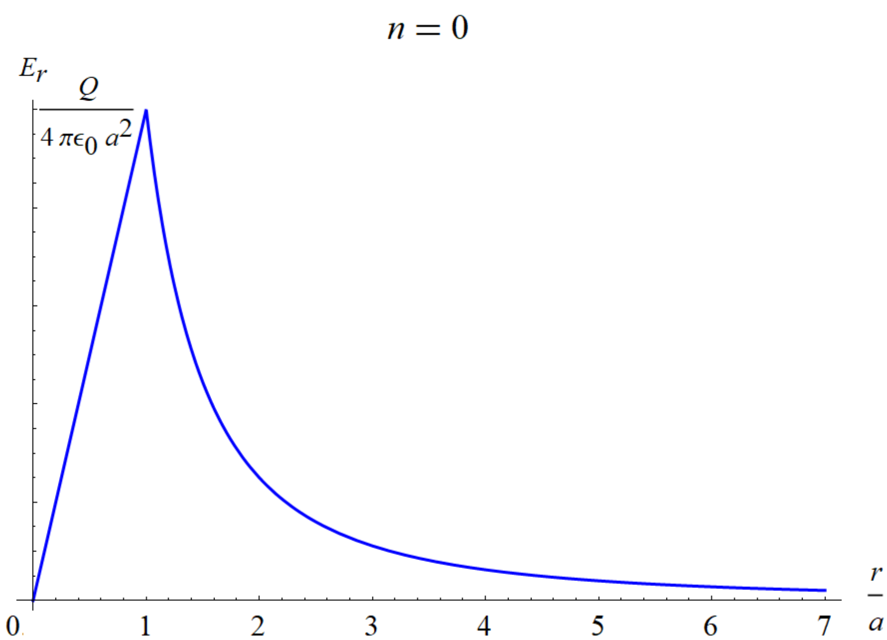
Below are plots of E_r versus r/a for $n = -2, \ldots, 2$ to illustrate the field's behavior.



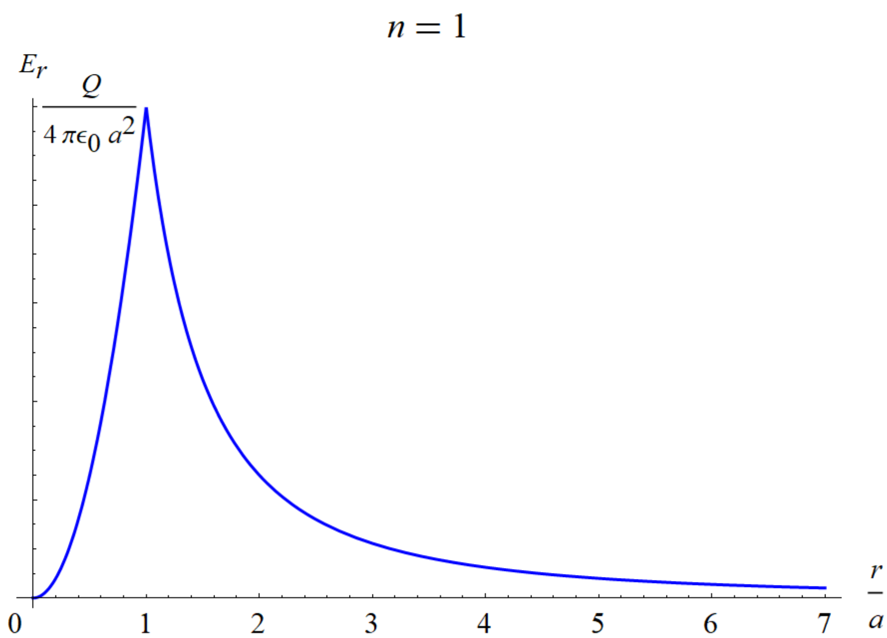
The field falls off as $1/r$ inside the sphere and falls off as $1/r^2$ outside of it.



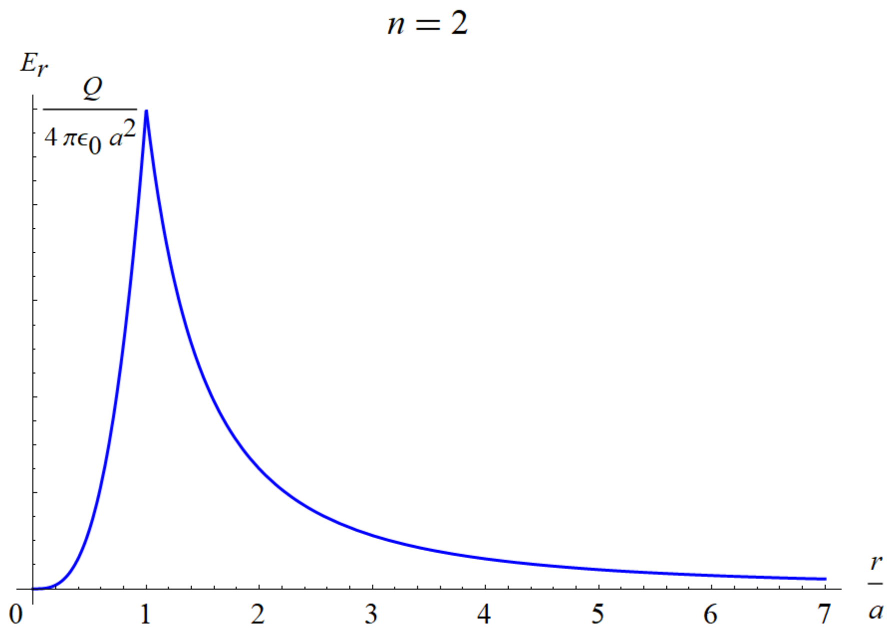
The field is constant inside the sphere and falls off as $1/r^2$ outside of it.



The field increases linearly inside the sphere and falls off as $1/r^2$ outside of it.



The field increases quadratically inside the sphere and falls off as $1/r^2$ outside of it.



The field increases cubically inside the sphere and falls off as $1/r^2$ outside of it.