According to Maxwell, there are four equations that govern the time evolution of the electric and magnetic fields, $\mathbf{E} = \mathbf{E}(x, y, z, t)$ and $\mathbf{B} = \mathbf{B}(x, y, z, t)$, in a vacuum.

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{J}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$\rho = \rho(x, y, z, t)$ and $\mathbf{J} = \mathbf{J}(x, y, z, t)$ are the charge density and the current density (also known as charge flux), respectively—both are given. The goal is to derive general formulas for $\mathbf{E}$ and $\mathbf{B}$ in all of space, that is, with no boundaries. Before starting, observe that taking the divergence of both sides of the first equation yields the continuity equation.

$$\nabla \cdot \left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) = \nabla \cdot (\nabla \times \mathbf{B} - \mu_0 \mathbf{J})$$

$$\frac{1}{c^2}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) = \underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{= \mathbf{0}} - \mu_0 \nabla \cdot \mathbf{J}$$

$$\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right) = -\mu_0 \nabla \cdot \mathbf{J}$$

$$\frac{1}{c^2 \epsilon_0 \mu_0}\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

Since $c^2 \epsilon_0 \mu_0 = 1$, we have

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}.}$$

This is a solvability condition: In order for $\mathbf{E}$ and $\mathbf{B}$ to exist, ρ and $\mathbf{J}$ must satisfy the continuity equation. In other words, ρ and $\mathbf{J}$ can't both be arbitrary. To physically interpret this equation, integrate both sides over a closed three-dimensional volume.

$$\iiint_V \frac{\partial \rho}{\partial t}\, dV = -\iiint_V \nabla \cdot \mathbf{J}\, dV$$

Apply the divergence theorem on the right side.

$$\frac{d}{dt}\iiint_V \rho\, dV = -\iint_S \mathbf{J} \cdot d\mathbf{S}$$

The rate of accumulation of charge in a volume must therefore be equal to the net rate of charge influx across the boundary.

Repeat Maxwell's equations.

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \left(\nabla \times \mathbf{B} - \mu_0 \mathbf{J}\right)$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \cdot \mathbf{B} = 0$$

Differentiate both sides of the first two equations with respect to time.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right) &= \frac{\partial}{\partial t}\left[c^2 \left(\nabla \times \mathbf{B} - \mu_0 \mathbf{J}\right)\right] \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c^2 \left(\nabla \times \frac{\partial \mathbf{B}}{\partial t} - \mu_0 \frac{\partial \mathbf{J}}{\partial t}\right) \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c^2 \left[\nabla \times (-\nabla \times \mathbf{E}) - \mu_0 \frac{\partial \mathbf{J}}{\partial t}\right] \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2 \nabla \times (\nabla \times \mathbf{E}) - c^2 \mu_0 \frac{\partial \mathbf{J}}{\partial t} \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2 [\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}] - \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}}{\partial t} \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2 \left[\nabla\left(\frac{\rho}{\epsilon_0}\right) - \nabla^2 \mathbf{E}\right] - \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}}{\partial t} \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c^2 \nabla^2 \mathbf{E} - \frac{c^2}{\epsilon_0}\nabla\rho - \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}}{\partial t}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{B}}{\partial t}\right) &= \frac{\partial}{\partial t}(-\nabla \times \mathbf{E}) \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -\nabla \times \frac{\partial \mathbf{E}}{\partial t} \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -\nabla \times \left[c^2 \left(\nabla \times \mathbf{B} - \mu_0 \mathbf{J}\right)\right] \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2 \nabla \times (\nabla \times \mathbf{B}) + c^2 \mu_0 \nabla \times \mathbf{J} \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2 [\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}] + c^2 \mu_0 \nabla \times \mathbf{J} \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2 [\nabla(0) - \nabla^2 \mathbf{B}] + c^2 \mu_0 \nabla \times \mathbf{J} \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= c^2 \nabla^2 \mathbf{B} + c^2 \mu_0 \nabla \times \mathbf{J}
\end{aligned}$$

We find that the electric and magnetic fields satisfy their own inhomogeneous wave equation. These will be solved in all of space for $t > 0$ with two prescribed initial conditions.

$$\mathbf{E}(x, y, z, 0) = \mathbf{E}_0(x, y, z) \qquad \mathbf{B}(x, y, z, 0) = \mathbf{B}_0(x, y, z)$$
$$\frac{\partial \mathbf{E}}{\partial t}(x, y, z, 0) = \mathbf{E}_0'(x, y, z) \qquad \frac{\partial \mathbf{B}}{\partial t}(x, y, z, 0) = \mathbf{B}_0'(x, y, z)$$

Because these wave equations are linear, the general solutions for $\mathbf{E}$ and $\mathbf{B}$ each can be written as the sum of a complementary solution and a particular solution.

$$\mathbf{E} = \mathbf{E}_c + \mathbf{E}_p$$
$$\mathbf{B} = \mathbf{B}_c + \mathbf{B}_p$$

The complementary solutions satisfy the associated homogeneous wave equations with nonzero initial conditions.

$$\begin{aligned}
&\frac{\partial^2 \mathbf{E}_c}{\partial t^2} = c^2 \nabla^2 \mathbf{E}_c, \quad -\infty < x, y, z < \infty,\ t > 0 \\
&\mathbf{E}_c(x, y, z, 0) = \mathbf{E}_0(x, y, z) \\
&\frac{\partial \mathbf{E}_c}{\partial t}(x, y, z, 0) = \mathbf{E}_0' = c^2 (\nabla \times \mathbf{B}_0)(x, y, z)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2 \mathbf{B}_c}{\partial t^2} = c^2 \nabla^2 \mathbf{B}_c, \quad -\infty < x, y, z < \infty,\ t > 0 \\
&\mathbf{B}_c(x, y, z, 0) = \mathbf{B}_0(x, y, z) \\
&\frac{\partial \mathbf{B}_c}{\partial t}(x, y, z, 0) = \mathbf{B}_0' = -(\nabla \times \mathbf{E}_0)(x, y, z)
\end{aligned}$$

The Complementary Solutions

The general solution to the homogeneous wave equation in space is given by Kirchhoff and Poisson's formula (see Appendix A).

$$\mathbf{E}_c(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \mathbf{E}_0(x_0,y_0,z_0)\,dS_0\right] + \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} [c^2(\nabla\times\mathbf{B}_0)(x_0,y_0,z_0)]\,dS_0$$

$$\mathbf{B}_c(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \mathbf{B}_0(x_0,y_0,z_0)\,dS_0\right] + \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} [-(\nabla\times\mathbf{E}_0)(x_0,y_0,z_0)]\,dS_0$$

These surface integrals are over the sphere in $x_0y_0z_0$-space centered at (x,y,z) with radius ct. Write them explicitly by using spherical coordinates (r_0,ϕ_0,θ_0), where r_0 is the radial distance from this sphere's center, ϕ_0 is the azimuthal angle, and θ_0 is the angle from the polar axis.

$$\begin{aligned} x_0 - x &= ct\sin\theta_0\cos\phi_0 \\ y_0 - y &= ct\sin\theta_0\sin\phi_0 \\ z_0 - z &= ct\cos\theta_0 \end{aligned}$$

As a result,

$$\mathbf{E}_c(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_0^{\pi}\int_0^{2\pi} \mathbf{E}_0(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0)\right] + \frac{1}{4\pi t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} (\nabla\times\mathbf{B}_0)(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}_c(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_0^{\pi}\int_0^{2\pi} \mathbf{B}_0(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0)\right] - \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} (\nabla\times\mathbf{E}_0)(x_0,y_0,z_0)\,dS_0.$$

Bring c^2t^2 out in front.

$$\mathbf{E}_c(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{E}_0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0\right] + \frac{1}{4\pi t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}(\nabla\times\mathbf{B}_0)(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}_c(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{B}_0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0\right] - \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}(\nabla\times\mathbf{E}_0)(x_0,y_0,z_0)\,dS_0$$

Evaluate the time derivatives by using the product and chain rules.

$$\begin{aligned}\mathbf{E}_c(x,y,z,t) = \left[\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{E}_0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 + \frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\partial\mathbf{E}_0}{\partial r_0}(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct)\sin\theta_0\,d\phi_0\,d\theta_0\right]\\ + \frac{1}{4\pi t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}(\nabla\times\mathbf{B}_0)(x_0,y_0,z_0)\,dS_0\end{aligned}$$

$$\begin{aligned}\mathbf{B}_c(x,y,z,t) = \left[\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{B}_0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 + \frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\partial\mathbf{B}_0}{\partial r_0}(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct)\sin\theta_0\,d\phi_0\,d\theta_0\right]\\ - \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}(\nabla\times\mathbf{E}_0)(x_0,y_0,z_0)\,dS_0\end{aligned}$$

Evaluate the remaining derivatives.

$$\mathbf{E}_c(x,y,z,t) = \left[\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{E}_0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 + \frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\partial\mathbf{E}_0}{\partial r_0}(ct,\phi_0,\theta_0)(c)\sin\theta_0\,d\phi_0\,d\theta_0\right] + \frac{1}{4\pi t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}(\nabla\times\mathbf{B}_0)(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}_c(x,y,z,t) = \left[\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{B}_0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 + \frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\partial\mathbf{B}_0}{\partial r_0}(ct,\phi_0,\theta_0)(c)\sin\theta_0\,d\phi_0\,d\theta_0\right] - \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}(\nabla\times\mathbf{E}_0)(x_0,y_0,z_0)\,dS_0$$

Prepare to write the differential as dS_0 again.

$$\mathbf{E}_c(x,y,z,t) = \frac{1}{4\pi t}\left[\frac{1}{c^2 t}\int_0^{\pi}\int_0^{2\pi}\mathbf{E}_0(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0) + \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\frac{\partial\mathbf{E}_0}{\partial r_0}(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0)\right] + \frac{1}{4\pi t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}(\nabla\times\mathbf{B}_0)(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}_c(x,y,z,t) = \frac{1}{4\pi c^2 t}\left[\frac{1}{t}\int_0^{\pi}\int_0^{2\pi}\mathbf{B}_0(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0) + c\int_0^{\pi}\int_0^{2\pi}\frac{\partial\mathbf{B}_0}{\partial r_0}(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0)\right] - \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}(\nabla\times\mathbf{E}_0)(x_0,y_0,z_0)\,dS_0$$

Write the integrals in square brackets over the sphere centered at (x, y, z) with radius ct.

$$\mathbf{E}_c(x,y,z,t) = \frac{1}{4\pi c^2 t}\left[\frac{1}{t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \mathbf{E}_0(ct,\phi_0,\theta_0)\,dS_0 + c\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \frac{\partial \mathbf{E}_0}{\partial r_0}(ct,\phi_0,\theta_0)\,dS_0\right] + \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} c^2(\nabla\times\mathbf{B}_0)(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}_c(x,y,z,t) = \frac{1}{4\pi c^2 t}\left[\frac{1}{t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \mathbf{B}_0(ct,\phi_0,\theta_0)\,dS_0 + c\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \frac{\partial \mathbf{B}_0}{\partial r_0}(ct,\phi_0,\theta_0)\,dS_0\right] - \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} (\nabla\times\mathbf{E}_0)(x_0,y_0,z_0)\,dS_0$$

Therefore, combining the integrals, the complementary solutions are obtained.

$$\mathbf{E}_c(x,y,z,t) = \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \left(\frac{1}{t}\mathbf{E}_0 + c\frac{\partial \mathbf{E}_0}{\partial r_0} + c^2\nabla\times\mathbf{B}_0\right) dS_0 = \frac{1}{4\pi c^2 t}\iint\limits_{r_0=ct}\left(\frac{1}{t}\mathbf{E}_0 + c\frac{\partial \mathbf{E}_0}{\partial r_0} + c^2\nabla\times\mathbf{B}_0\right) dS_0$$

$$\mathbf{B}_c(x,y,z,t) = \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \left(\frac{1}{t}\mathbf{B}_0 + c\frac{\partial \mathbf{B}_0}{\partial r_0} - \nabla\times\mathbf{E}_0\right) dS_0 = \frac{1}{4\pi c^2 t}\iint\limits_{r_0=ct}\left(\frac{1}{t}\mathbf{B}_0 + c\frac{\partial \mathbf{B}_0}{\partial r_0} - \nabla\times\mathbf{E}_0\right) dS_0$$

The Particular Solutions

On the other hand, the particular solutions satisfy the inhomogeneous wave equations with zero initial conditions.

$$\frac{\partial^2 \mathbf{E}_p}{\partial t^2} = c^2\nabla^2\mathbf{E}_p - \frac{c^2}{\epsilon_0}\nabla\rho - \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}}{\partial t}, \quad -\infty < x, y, z < \infty,\ t > 0$$

$$\mathbf{E}_p(x, y, z, 0) = \mathbf{0}$$

$$\frac{\partial \mathbf{E}_p}{\partial t}(x, y, z, 0) = \mathbf{0}$$

$$\frac{\partial^2 \mathbf{B}_p}{\partial t^2} = c^2\nabla^2\mathbf{B}_p + c^2\mu_0\nabla\times\mathbf{J}, \quad -\infty < x, y, z < \infty,\ t > 0$$

$$\mathbf{B}_p(x, y, z, 0) = \mathbf{0}$$

$$\frac{\partial \mathbf{B}_p}{\partial t}(x, y, z, 0) = \mathbf{0}$$

By Duhamel's principle, the solutions to these inhomogeneous wave equations are

$$\mathbf{E}_p(x, y, z, t) = \int_0^t \mathbf{E}_h(x, y, z, t-s; s)\,ds \quad \text{and} \quad \mathbf{B}_p(x, y, z, t) = \int_0^t \mathbf{B}_h(x, y, z, t-s; s)\,ds,$$

where $\mathbf{E}_h(x, y, z, t; s)$ and $\mathbf{B}_h(x, y, z, t; s)$ are solutions to the associated homogeneous equations with specific initial conditions.

$$\frac{\partial^2 \mathbf{E}_h}{\partial t^2} = c^2\nabla^2\mathbf{E}_h, \quad -\infty < x, y, z < \infty,\ t > 0$$

$$\mathbf{E}_h(x, y, z, 0; s) = \mathbf{0}$$

$$\frac{\partial \mathbf{E}_h}{\partial t}(x, y, z, 0; s) = -\frac{c^2}{\epsilon_0}(\nabla\rho)(x, y, z, s) - \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}}{\partial t}(x, y, z, s)$$

$$\frac{\partial^2 \mathbf{B}_h}{\partial t^2} = c^2\nabla^2\mathbf{B}_h, \quad -\infty < x, y, z < \infty,\ t > 0$$

$$\mathbf{B}_h(x, y, z, 0; s) = \mathbf{0}$$

$$\frac{\partial \mathbf{B}_h}{\partial t}(x, y, z, 0; s) = c^2\mu_0(\nabla\times\mathbf{J})(x, y, z, s)$$

The solutions for $\mathbf{E}_h$ and $\mathbf{B}_h$ are given by Kirchhoff and Poisson's formula.

$$\mathbf{E}_h(x, y, z, t; s) = \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \left[-\frac{c^2}{\epsilon_0}(\nabla_0\rho)(x_0, y_0, z_0, s) - \frac{1}{\epsilon_0}\frac{\partial \mathbf{J}}{\partial t}(x_0, y_0, z_0, s)\right] dS_0$$

$$\mathbf{B}_h(x, y, z, t; s) = \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} [c^2\mu_0(\nabla_0\times\mathbf{J})(x_0, y_0, z_0, s)]\, dS_0$$

Bring the constants in front.

$$\mathbf{E}_h(x,y,z,t;s) = -\frac{1}{4\pi\epsilon_0 c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \left[\frac{\partial \mathbf{J}}{\partial t}(x_0,y_0,z_0,s) + c^2(\nabla_0\rho)(x_0,y_0,z_0,s)\right] dS_0$$

$$\mathbf{B}_h(x,y,z,t;s) = \frac{\mu_0}{4\pi t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} (\nabla_0 \times \mathbf{J})(x_0,y_0,z_0,s)\, dS_0$$

Now $\mathbf{E}_p$ and $\mathbf{B}_p$ can be determined. Do the calculation for $\mathbf{E}_p$ first.

$$\begin{aligned}
\mathbf{E}_p(x,y,z,t) &= \int_0^t \mathbf{E}_h(x,y,z,t-s;s)\, ds \\
&= \int_0^t \left\{ -\frac{1}{4\pi\epsilon_0 c^2 (t-s)} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2(t-s)^2}} \left[\frac{\partial \mathbf{J}}{\partial t}(x_0,y_0,z_0,s) + c^2(\nabla_0\rho)(x_0,y_0,z_0,s)\right] dS_0 \right\} ds \\
&= -\frac{1}{4\pi\epsilon_0 c^2} \int_0^t \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2(t-s)^2}} \frac{1}{t-s} \left[\frac{\partial \mathbf{J}}{\partial t}(x_0,y_0,z_0,s) + c^2(\nabla_0\rho)(x_0,y_0,z_0,s)\right] dS_0\, ds
\end{aligned}$$

Write the surface integral explicitly by using spherical coordinates (r_0, ϕ_0, θ_0).

$$\begin{aligned}
x_0 - x &= c(t-s)\sin\theta_0\cos\phi_0 \\
y_0 - y &= c(t-s)\sin\theta_0\sin\phi_0 \\
z_0 - z &= c(t-s)\cos\theta_0
\end{aligned}$$

The solution becomes

$$\begin{aligned}
\mathbf{E}_p(x,y,z,t) &= -\frac{1}{4\pi\epsilon_0 c^2} \int_0^t \int_0^\pi \int_0^{2\pi} \frac{1}{t-s} \left[\frac{\partial \mathbf{J}}{\partial t}(c(t-s),\phi_0,\theta_0,s) + c^2(\nabla_0\rho)(c(t-s),\phi_0,\theta_0,s)\right] c^2(t-s)^2 \sin\theta_0\, d\phi_0\, d\theta_0\, ds \\
&= \frac{1}{4\pi\epsilon_0 c^2} \int_0^t \int_0^\pi \int_0^{2\pi} \left[\frac{\partial \mathbf{J}}{\partial t}(c(t-s),\phi_0,\theta_0,s) + c^2(\nabla_0\rho)(c(t-s),\phi_0,\theta_0,s)\right] c(t-s)\sin\theta_0\, d\phi_0\, d\theta_0\, (-c\, ds).
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
r_0 &= c(t-s) \quad \to \quad s = t - \frac{r_0}{c} \\
dr_0 &= -c\, ds
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathbf{E}_p(x,y,z,t) &= \frac{1}{4\pi\epsilon_0 c^2} \int_{ct}^0 \int_0^\pi \int_0^{2\pi} \left[\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) + c^2(\nabla_0\rho)\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0\sin\theta_0\, d\phi_0\, d\theta_0\, dr_0 \\
&= \frac{1}{4\pi\epsilon_0} \int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) - (\nabla_0\rho)\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0\sin\theta_0\, dr_0\, d\phi_0\, d\theta_0.
\end{aligned}$$

Consider the gradient in $x_0y_0z_0$-space of $\rho(r_0, \phi_0, \theta_0, t - r_0/c)$.

$$\begin{aligned}
\nabla_0\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) &= \left(\hat{\mathbf{r}}_0\frac{\partial}{\partial r_0}+\frac{\hat{\boldsymbol{\phi}}_0}{r_0\sin\theta_0}\frac{\partial}{\partial\phi_0}+\frac{\hat{\boldsymbol{\theta}}_0}{r_0}\frac{\partial}{\partial\theta_0}\right)\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) \\
&= \hat{\mathbf{r}}_0\left[\frac{\partial\rho}{\partial r_0}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+\frac{\partial\rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\frac{\partial}{\partial r_0}\left(t-\frac{r_0}{c}\right)\right] \\
&\qquad + \frac{\hat{\boldsymbol{\phi}}_0}{r_0\sin\theta_0}\frac{\partial\rho}{\partial\phi_0}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) \\
&\qquad\qquad + \frac{\hat{\boldsymbol{\theta}}_0}{r_0}\frac{\partial\rho}{\partial\theta_0}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) \\
&= \hat{\mathbf{r}}_0\left[\frac{\partial\rho}{\partial r_0}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)-\frac{1}{c}\frac{\partial\rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] \\
&\qquad + \frac{\hat{\boldsymbol{\phi}}_0}{r_0\sin\theta_0}\frac{\partial\rho}{\partial\phi_0}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) \\
&\qquad\qquad + \frac{\hat{\boldsymbol{\theta}}_0}{r_0}\frac{\partial\rho}{\partial\theta_0}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) \\
&= \hat{\mathbf{r}}_0\frac{\partial\rho}{\partial r_0}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+\frac{\hat{\boldsymbol{\phi}}_0}{r_0\sin\theta_0}\frac{\partial\rho}{\partial\phi_0}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) \\
&\qquad + \frac{\hat{\boldsymbol{\theta}}_0}{r_0}\frac{\partial\rho}{\partial\theta_0}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) \\
&= (\nabla_0\rho)\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)
\end{aligned}$$

That means

$$(\nabla_0\rho)\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)=\nabla_0\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right).$$

Noting the following unit vector derivatives,

$$\frac{\partial}{\partial r_0}\hat{\mathbf{r}}_0 = \mathbf{0} \quad \text{and} \quad \frac{\partial}{\partial \phi_0}\hat{\boldsymbol{\phi}}_0 = -\hat{\mathbf{r}}_0 \sin\theta_0 - \hat{\boldsymbol{\theta}}_0 \cos\theta_0 \quad \text{and} \quad \frac{\partial}{\partial \theta_0}\hat{\boldsymbol{\theta}}_0 = -\hat{\mathbf{r}}_0,$$

substitute this formula for $(\nabla_0 \rho)$ into the one for $\mathbf{E}_p$.

$$\begin{aligned}
\mathbf{E}_p(x,y,z,t) &= \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) - (\nabla_0\rho)\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\theta_0\, dr_0\, d\phi_0\, d\theta_0 \\
&= \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial \rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\theta_0\, dr_0\, d\phi_0\, d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\hat{\mathbf{r}}_0 \frac{\partial}{\partial r_0}\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) + \frac{\hat{\boldsymbol{\phi}}_0}{r_0\sin\theta_0}\frac{\partial}{\partial \phi_0}\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) + \frac{\hat{\boldsymbol{\theta}}_0}{r_0}\frac{\partial}{\partial \theta_0}\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\theta_0\, dr_0\, d\phi_0\, d\theta_0 \\
&= \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial \rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\theta_0\, dr_0\, d\phi_0\, d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \hat{\mathbf{r}}_0 r_0 \frac{\partial}{\partial r_0}\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) \sin\theta_0\, dr_0\, d\phi_0\, d\theta_0 \\
&\qquad - \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \hat{\boldsymbol{\phi}}_0 \frac{\partial}{\partial \phi_0}\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) dr_0\, d\phi_0\, d\theta_0 \\
&\qquad\quad - \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \hat{\boldsymbol{\theta}}_0 \sin\theta_0 \frac{\partial}{\partial \theta_0}\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) dr_0\, d\phi_0\, d\theta_0 \\
&= \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial \rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\theta_0\, dr_0\, d\phi_0\, d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{2\pi} \left[\int_0^{ct} \hat{\mathbf{r}}_0 r_0 \frac{\partial}{\partial r_0}\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) dr_0\right] \sin\theta_0\, d\phi_0\, d\theta_0 \\
&\qquad - \frac{1}{4\pi\epsilon_0}\int_0^\pi \int_0^{ct} \left[\int_0^{2\pi} \hat{\boldsymbol{\phi}}_0 \frac{\partial}{\partial \phi_0}\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) d\phi_0\right] dr_0\, d\theta_0 \\
&\qquad\quad - \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \int_0^{ct} \left[\int_0^\pi \hat{\boldsymbol{\theta}}_0 \sin\theta_0 \frac{\partial}{\partial \theta_0}\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) d\theta_0\right] dr_0\, d\phi_0
\end{aligned}$$

Evaluate the integrals in square brackets by parts.

$$
\begin{aligned}
\mathbf{E}_p(x,y,z,t) &= \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)-\frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\left[\hat{\mathbf{r}}_0\,r_0\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\bigg|_0^{ct}-\int_0^{ct}\frac{\partial}{\partial r_0}(\hat{\mathbf{r}}_0 r_0)\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)dr_0\right]\sin\theta_0\,d\phi_0\,d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{ct}\left[\hat{\boldsymbol{\phi}}_0\,\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\bigg|_0^{2\pi}-\int_0^{2\pi}\frac{\partial}{\partial\phi_0}(\hat{\boldsymbol{\phi}}_0)\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)d\phi_0\right]dr_0\,d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{ct}\left[\hat{\boldsymbol{\theta}}_0\sin\theta_0\,\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\bigg|_0^{\pi}-\int_0^{\pi}\frac{\partial}{\partial\theta_0}(\hat{\boldsymbol{\theta}}_0\sin\theta_0)\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)d\theta_0\right]dr_0\,d\phi_0 \\
&= \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)-\frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\left[\hat{\mathbf{r}}_0\,ct\,\rho(ct,\phi_0,\theta_0,0)-0\right]\sin\theta_0\,d\phi_0\,d\theta_0 \\
&\quad + \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\hat{\mathbf{r}}_0\,\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)dr_0\,\sin\theta_0\,d\phi_0\,d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{ct}\left[\cancel{\hat{\boldsymbol{\phi}}_0\,\rho\left(r_0,2\pi,\theta_0,t-\frac{r_0}{c}\right)-\hat{\boldsymbol{\phi}}_0\,\rho\left(r_0,0,\theta_0,t-\frac{r_0}{c}\right)}\right]dr_0\,d\theta_0 \\
&\quad + \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{ct}\int_0^{2\pi}(-\hat{\mathbf{r}}_0\sin\theta_0-\cancel{\hat{\boldsymbol{\theta}}_0\cos\theta_0})\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)d\phi_0\,dr_0\,d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{ct}\left[\hat{\boldsymbol{\theta}}_0\,\rho\cancel{\left(r_0,\phi_0,\pi,t-\frac{r_0}{c}\right)}\sin\pi-\hat{\boldsymbol{\theta}}_0\,\rho\cancel{\left(r_0,\phi_0,0,t-\frac{r_0}{c}\right)}\sin 0\right]dr_0\,d\phi_0 \\
&\quad + \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{ct}\int_0^{\pi}(-\hat{\mathbf{r}}_0\sin\theta_0+\cancel{\hat{\boldsymbol{\theta}}_0\cos\theta_0})\rho\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)d\theta_0\,dr_0\,d\phi_0
\end{aligned}
$$

Finally, use the fact that $\nabla \cdot \mathbf{E}_p(x, y, z, 0) = \rho(x, y, z, 0)/\epsilon_0 = 0$.

$$
\begin{aligned}
\mathbf{E}_p(x, y, z, t) &= \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\hat{\mathbf{r}}_0 \, ct \, \rho(ct, \phi_0, \theta_0, 0) \sin\theta_0 \, d\phi_0 \, d\theta_0 \\
&\quad - \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\hat{\mathbf{r}}_0 \, \rho\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) dr_0 \, \sin\theta_0 \, d\phi_0 \, d\theta_0 \\
&= \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[-\frac{\hat{\mathbf{r}}_0}{r_0^2}\rho\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) - \frac{\hat{\mathbf{r}}_0}{cr_0}\frac{\partial\rho}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) - \frac{1}{c^2 r_0}\frac{\partial\mathbf{J}}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right)\right] r_0^2 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&= \frac{1}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[-\frac{\mathbf{r}_0}{r_0^3}\rho\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) - \frac{\mathbf{r}_0}{cr_0^2}\frac{\partial\rho}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) - \frac{1}{c^2 r_0}\frac{\partial\mathbf{J}}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right)\right] r_0^2 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&= -\frac{1}{4\pi\epsilon_0}\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\le c^2t^2}}\left(\frac{\mathbf{r}_0}{r_0^3}\rho + \frac{\mathbf{r}_0}{cr_0^2}\frac{\partial\rho}{\partial t} + \frac{1}{c^2 r_0}\frac{\partial\mathbf{J}}{\partial t}\right)\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) dV_0 \\
&= -\frac{1}{4\pi\epsilon_0}\iiint\limits_{r_0\le ct}\left(\frac{\mathbf{r}_0}{r_0^3}\rho + \frac{\mathbf{r}_0}{cr_0^2}\frac{\partial\rho}{\partial t} + \frac{1}{c^2 r_0}\frac{\partial\mathbf{J}}{\partial t}\right)\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) dV_0
\end{aligned}
$$

In Cartesian coordinates ($dV_0 = dx_0 \, dy_0 \, dz_0$) this is written as

$$
\begin{aligned}
\mathbf{E}_p(x, y, z, t) = \frac{1}{4\pi\epsilon_0}\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\le c^2t^2}} &\left\{\frac{(x-x_0)\hat{\mathbf{x}} + (y-y_0)\hat{\mathbf{y}} + (z-z_0)\hat{\mathbf{z}}}{\sqrt{[(x_0-x)^2+(y_0-y)^2+(z_0-z)^2]^3}}\,\rho\left(x_0, y_0, z_0, t - \frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right)\right. \\
&+ \frac{1}{c}\frac{(x-x_0)\hat{\mathbf{x}} + (y-y_0)\hat{\mathbf{y}} + (z-z_0)\hat{\mathbf{z}}}{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}\frac{\partial\rho}{\partial t}\left(x_0, y_0, z_0, t - \frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right) \\
&\left. - \frac{1}{c^2}\frac{1}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}\frac{\partial\mathbf{J}}{\partial t}\left(x_0, y_0, z_0, t - \frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right)\right\} dV_0.
\end{aligned}
$$

Turning our attention back to the magnetic field, plug the formula for $\mathbf{B}_h$ into the one for $\mathbf{B}_p$.

$$\begin{aligned}\mathbf{B}_p(x,y,z,t) &= \int_0^t \mathbf{B}_h(x,y,z,t-s;s)\,ds \\ &= \int_0^t \left\{\frac{\mu_0}{4\pi(t-s)} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2(t-s)^2}} (\nabla_0\times\mathbf{J})(x_0,y_0,z_0,s)\,dS_0\right\} ds \\ &= \frac{\mu_0}{4\pi}\int_0^t \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2(t-s)^2}} \frac{1}{t-s}\left[(\nabla_0\times\mathbf{J})(x_0,y_0,z_0,s)\right] dS_0\,ds\end{aligned}$$

Write the surface integral over the sphere centered at (x,y,z) with radius $c(t-s)$ explicitly by using spherical coordinates (r_0,ϕ_0,θ_0).

$$\begin{aligned}x_0-x &= c(t-s)\sin\theta_0\cos\phi_0 \\ y_0-y &= c(t-s)\sin\theta_0\sin\phi_0 \\ z_0-z &= c(t-s)\cos\theta_0\end{aligned}$$

The solution becomes

$$\begin{aligned}\mathbf{B}_p(x,y,z,t) &= \frac{\mu_0}{4\pi}\int_0^t\int_0^\pi\int_0^{2\pi}\frac{1}{t-s}\left[(\nabla_0\times\mathbf{J})(c(t-s),\phi_0,\theta_0,s)\right] c^2(t-s)^2\sin\theta_0\,d\phi_0\,d\theta_0\,ds \\ &= -\frac{\mu_0}{4\pi}\int_0^t\int_0^\pi\int_0^{2\pi}\left[(\nabla_0\times\mathbf{J})(c(t-s),\phi_0,\theta_0,s)\right] c(t-s)\sin\theta_0\,d\phi_0\,d\theta_0\,(-c\,ds).\end{aligned}$$

Make the following substitution.

$$\begin{aligned}r_0 &= c(t-s) \quad\to\quad s = t-\frac{r_0}{c} \\ dr_0 &= -c\,ds\end{aligned}$$

As a result,

$$\begin{aligned}\mathbf{B}_p(x,y,z,t) &= -\frac{\mu_0}{4\pi}\int_{ct}^0\int_0^\pi\int_0^{2\pi}\left[(\nabla_0\times\mathbf{J})\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0\sin\theta_0\,d\phi_0\,d\theta_0\,dr_0 \\ &= \frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[(\nabla_0\times\mathbf{J})\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0.\end{aligned}$$

Consider the curl in $x_0y_0z_0$-space of $\mathbf{J}(r_0,\phi_0,\theta_0,t-r_0/c)$.

$$\begin{aligned}\left(\hat{\mathbf{r}}_0\frac{\partial}{\partial r_0}+\frac{\hat{\boldsymbol{\phi}}_0}{r_0\sin\theta_0}\frac{\partial}{\partial\phi_0}+\frac{\hat{\boldsymbol{\theta}}_0}{r_0}\frac{\partial}{\partial\theta_0}\right)\times\mathbf{J}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) &= \left[\frac{1}{r_0\sin\theta_0}\frac{\partial}{\partial\theta_0}(J_\phi\sin\theta_0)-\frac{1}{r_0\sin\theta_0}\frac{\partial J_\theta}{\partial\phi_0}\right]\hat{\mathbf{r}}_0 \\ &\quad+\left[\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0J_\theta)-\frac{1}{r_0}\frac{\partial J_r}{\partial\theta_0}\right]\hat{\boldsymbol{\phi}}_0 \\ &\quad+\left[\frac{1}{r_0\sin\theta_0}\frac{\partial J_r}{\partial\phi_0}-\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0J_\phi)\right]\hat{\boldsymbol{\theta}}_0\end{aligned}$$

The angular derivatives are unaffected, but the radial derivative changes.

$$
\begin{aligned}
\left(\hat{\mathbf{r}}_0\frac{\partial}{\partial r_0}+\frac{\hat{\boldsymbol{\phi}}_0}{r_0\sin\theta_0}\frac{\partial}{\partial\phi_0}+\frac{\hat{\boldsymbol{\theta}}_0}{r_0}\frac{\partial}{\partial\theta_0}\right)\times\mathbf{J}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right) &= \left[\frac{1}{r_0\sin\theta_0}\frac{\partial}{\partial\theta_0}(J_\phi\sin\theta_0)-\frac{1}{r_0\sin\theta_0}\frac{\partial J_\theta}{\partial\phi_0}\right]\hat{\mathbf{r}}_0 \\
&\quad+\left\{\frac{1}{r_0}\left[J_\theta+r_0\left[\frac{\partial J_\theta}{\partial r_0}+\frac{\partial J_\theta}{\partial t}\frac{\partial}{\partial r_0}\left(t-\frac{r_0}{c}\right)\right]\right]-\frac{1}{r_0}\frac{\partial J_r}{\partial\theta_0}\right\}\hat{\boldsymbol{\phi}}_0 \\
&\quad+\left\{\frac{1}{r_0\sin\theta_0}\frac{\partial J_r}{\partial\phi_0}-\frac{1}{r_0}\left[J_\phi+r_0\left[\frac{\partial J_\phi}{\partial r_0}+\frac{\partial J_\phi}{\partial t}\frac{\partial}{\partial r_0}\left(t-\frac{r_0}{c}\right)\right]\right]\right\}\hat{\boldsymbol{\theta}}_0 \\
&= \left[\frac{1}{r_0\sin\theta_0}\frac{\partial}{\partial\theta_0}(J_\phi\sin\theta_0)-\frac{1}{r_0\sin\theta_0}\frac{\partial J_\theta}{\partial\phi_0}\right]\hat{\mathbf{r}}_0 \\
&\quad+\left\{\frac{1}{r_0}\left[J_\theta+r_0\left(\frac{\partial J_\theta}{\partial r_0}-\frac{1}{c}\frac{\partial J_\theta}{\partial t}\right)\right]-\frac{1}{r_0}\frac{\partial J_r}{\partial\theta_0}\right\}\hat{\boldsymbol{\phi}}_0 \\
&\quad+\left\{\frac{1}{r_0\sin\theta_0}\frac{\partial J_r}{\partial\phi_0}-\frac{1}{r_0}\left[J_\phi+r_0\left(\frac{\partial J_\phi}{\partial r_0}-\frac{1}{c}\frac{\partial J_\phi}{\partial t}\right)\right]\right\}\hat{\boldsymbol{\theta}}_0 \\
&= \left[\frac{1}{r_0\sin\theta_0}\frac{\partial}{\partial\theta_0}(J_\phi\sin\theta_0)-\frac{1}{r_0\sin\theta_0}\frac{\partial J_\theta}{\partial\phi_0}\right]\hat{\mathbf{r}}_0 \\
&\quad+\left\{\frac{1}{r_0}\left(J_\theta+r_0\frac{\partial J_\theta}{\partial r_0}\right)-\frac{1}{r_0}\frac{\partial J_r}{\partial\theta_0}\right\}\hat{\boldsymbol{\phi}}_0-\frac{1}{c}\frac{\partial J_\theta}{\partial t}\hat{\boldsymbol{\phi}}_0 \\
&\quad+\left\{\frac{1}{r_0\sin\theta_0}\frac{\partial J_r}{\partial\phi_0}-\frac{1}{r_0}\left(J_\phi+r_0\frac{\partial J_\phi}{\partial r_0}\right)\right\}\hat{\boldsymbol{\theta}}_0+\frac{1}{c}\frac{\partial J_\phi}{\partial t}\hat{\boldsymbol{\theta}}_0 \\
&= (\nabla_0\times\mathbf{J})\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+\frac{1}{c}\left[\frac{\partial J_\phi}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\hat{\boldsymbol{\theta}}_0-\frac{\partial J_\theta}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\hat{\boldsymbol{\phi}}_0\right] \\
&= (\nabla_0\times\mathbf{J})\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+\frac{1}{c}\left[-\frac{\partial J_r}{\partial t}(\hat{\mathbf{r}}_0\times\hat{\mathbf{r}}_0)+\frac{\partial J_\phi}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)(-\hat{\mathbf{r}}_0\times\hat{\boldsymbol{\phi}}_0)\right. \\
&\qquad\left.-\frac{\partial J_\theta}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)(\hat{\mathbf{r}}_0\times\hat{\boldsymbol{\theta}}_0)\right] \\
&= (\nabla_0\times\mathbf{J})\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)-\frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)
\end{aligned}
$$

That means

$$(\nabla_0 \times \mathbf{J})\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) = \left(\hat{\mathbf{r}}_0 \frac{\partial}{\partial r_0} + \frac{\hat{\boldsymbol{\phi}}_0}{r_0 \sin\theta_0}\frac{\partial}{\partial \phi_0} + \frac{\hat{\boldsymbol{\theta}}_0}{r_0}\frac{\partial}{\partial \theta_0}\right) \times \mathbf{J}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right) + \frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right).$$

Substitute this formula for $(\nabla_0 \times \mathbf{J})$ into the one for $\mathbf{B}_p$.

$$\begin{aligned}
\mathbf{B}_p(x,y,z,t) &= \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi}\int_0^{ct}\left[(\nabla_0 \times \mathbf{J})\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&= \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&\quad + \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi}\int_0^{ct}\left[\left(\hat{\mathbf{r}}_0 \frac{\partial}{\partial r_0} + \frac{\hat{\boldsymbol{\phi}}_0}{r_0 \sin\theta_0}\frac{\partial}{\partial \phi_0} + \frac{\hat{\boldsymbol{\theta}}_0}{r_0}\frac{\partial}{\partial \theta_0}\right) \times \mathbf{J}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&= \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&\quad + \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi}\int_0^{ct}\left\{\hat{\mathbf{r}}_0\left[\frac{1}{r_0\sin\theta_0}\frac{\partial}{\partial\theta_0}(J_\phi \sin\theta_0) - \frac{1}{r_0\sin\theta_0}\frac{\partial}{\partial\phi_0}J_\theta\right]\right. \\
&\qquad + \hat{\boldsymbol{\phi}}_0\left[\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0 J_\theta) - \frac{1}{r_0}\frac{\partial}{\partial\theta_0}J_r\right] \\
&\qquad \left. + \hat{\boldsymbol{\theta}}_0\left[\frac{1}{r_0\sin\theta_0}\frac{\partial}{\partial\phi_0}J_r - \frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0 J_\phi)\right]\right\} r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&= \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&\quad + \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi}\int_0^{ct}\left\{\hat{\mathbf{r}}_0\left[\frac{\partial}{\partial\theta_0}(J_\phi \sin\theta_0) - \frac{\partial}{\partial\phi_0}J_\theta\right]\right. \\
&\qquad + \hat{\boldsymbol{\phi}}_0\left[\sin\theta_0\frac{\partial}{\partial r_0}(r_0 J_\theta) - \sin\theta_0\frac{\partial}{\partial\theta_0}J_r\right] \\
&\qquad \left. + \hat{\boldsymbol{\theta}}_0\left[\frac{\partial}{\partial\phi_0}J_r - \sin\theta_0\frac{\partial}{\partial r_0}(r_0 J_\phi)\right]\right\} dr_0 \, d\phi_0 \, d\theta_0 \\
&= \frac{\mu_0}{4\pi}\left\{\int_0^\pi \int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0\right. \\
&\qquad + \int_0^{2\pi}\int_0^{ct}\left[\int_0^\pi \hat{\mathbf{r}}_0 \frac{\partial}{\partial\theta_0}(J_\phi\sin\theta_0)\, d\theta_0\right] dr_0\, d\phi_0 - \int_0^\pi\int_0^{ct}\left[\int_0^{2\pi}\hat{\mathbf{r}}_0\frac{\partial}{\partial\phi_0}J_\theta\, d\phi_0\right] dr_0\, d\theta_0 \\
&\qquad + \int_0^\pi\int_0^{2\pi}\left[\int_0^{ct}\hat{\boldsymbol{\phi}}_0\frac{\partial}{\partial r_0}(r_0 J_\theta)\, dr_0\right]\sin\theta_0\, d\phi_0\, d\theta_0 - \int_0^{2\pi}\int_0^{ct}\left[\int_0^\pi \hat{\boldsymbol{\phi}}_0 \sin\theta_0\frac{\partial}{\partial\theta_0}J_r\, d\theta_0\right] dr_0\, d\phi_0 \\
&\qquad \left. + \int_0^\pi\int_0^{ct}\left[\int_0^{2\pi}\hat{\boldsymbol{\theta}}_0\frac{\partial}{\partial\phi_0}J_r\, d\phi_0\right] dr_0\, d\theta_0 - \int_0^\pi\int_0^{2\pi}\left[\int_0^{ct}\hat{\boldsymbol{\theta}}_0\frac{\partial}{\partial r_0}(r_0 J_\phi)\, dr_0\right]\sin\theta_0\, d\phi_0\, d\theta_0\right\}
\end{aligned}$$

Evaluate the integrals in square brackets by parts.

$$\begin{aligned}
\mathbf{B}_p(x,y,z,t) = \frac{\mu_0}{4\pi}\Bigg\{ &\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&+ \int_0^{2\pi} \int_0^{ct} \left[\cancel{\hat{\mathbf{r}}_0 (J_\phi \sin\theta_0)\Big|_0^\pi} - \int_0^\pi (J_\phi \sin\theta_0)\frac{\partial}{\partial\theta_0}\hat{\mathbf{r}}_0 \, d\theta_0\right] dr_0 \, d\phi_0 \\
&- \int_0^\pi \int_0^{ct} \left[\cancel{\hat{\mathbf{r}}_0 J_\theta\Big|_0^{2\pi}} - \int_0^{2\pi} J_\theta \frac{\partial}{\partial\phi_0}\hat{\mathbf{r}}_0 \, d\phi_0\right] dr_0 \, d\theta_0 \\
&+ \int_0^\pi \int_0^{2\pi} \left[\hat{\boldsymbol{\phi}}_0 (r_0 J_\theta)\Big|_0^{ct} - \int_0^{ct} (r_0 J_\theta)\frac{\partial}{\partial r_0}\hat{\boldsymbol{\phi}}_0 \, dr_0\right] \sin\theta_0 \, d\phi_0 \, d\theta_0 \\
&- \int_0^{2\pi} \int_0^{ct} \left[\cancel{(\hat{\boldsymbol{\phi}}_0 \sin\theta_0) J_r\Big|_0^\pi} - \int_0^\pi J_r \frac{\partial}{\partial\theta_0}(\hat{\boldsymbol{\phi}}_0 \sin\theta_0) \, d\theta_0\right] dr_0 \, d\phi_0 \\
&+ \int_0^\pi \int_0^{ct} \left[\cancel{\hat{\boldsymbol{\theta}}_0 J_r\Big|_0^{2\pi}} - \int_0^{2\pi} J_r \frac{\partial}{\partial\phi_0}\hat{\boldsymbol{\theta}}_0 \, d\phi_0\right] dr_0 \, d\theta_0 \\
&- \int_0^\pi \int_0^{2\pi} \left[\hat{\boldsymbol{\theta}}_0 (r_0 J_\phi)\Big|_0^{ct} - \int_0^{ct} (r_0 J_\phi)\frac{\partial}{\partial r_0}\hat{\boldsymbol{\theta}}_0 \, dr_0\right] \sin\theta_0 \, d\phi_0 \, d\theta_0 \Bigg\} \\
= \frac{\mu_0}{4\pi}\Bigg\{ &\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&+ \int_0^{2\pi} \int_0^{ct} \left[-\int_0^\pi (J_\phi \sin\theta_0)\hat{\boldsymbol{\theta}}_0 \, d\theta_0\right] dr_0 \, d\phi_0 \\
&- \int_0^\pi \int_0^{ct} \left[-\int_0^{2\pi} J_\theta (\hat{\boldsymbol{\phi}}_0 \sin\theta_0) \, d\phi_0\right] dr_0 \, d\theta_0 \\
&+ \int_0^\pi \int_0^{2\pi} \left[\hat{\boldsymbol{\phi}}_0 \, ct \, J_\theta(ct,\phi_0,\theta_0,0) - \cancel{\int_0^{ct} (r_0 J_\theta)(0) \, dr_0}\right] \sin\theta_0 \, d\phi_0 \, d\theta_0 \\
&\cancel{- \int_0^{2\pi} \int_0^{ct} \left[-\int_0^\pi J_r (\hat{\boldsymbol{\phi}}_0 \cos\theta_0) \, d\theta_0\right] dr_0 \, d\phi_0} \\
&\cancel{+ \int_0^\pi \int_0^{ct} \left[-\int_0^{2\pi} J_r (\hat{\boldsymbol{\phi}}_0 \cos\theta_0) \, d\phi_0\right] dr_0 \, d\theta_0} \\
&- \int_0^\pi \int_0^{2\pi} \left[\hat{\boldsymbol{\theta}}_0 \, ct \, J_\phi(ct,\phi_0,\theta_0,0) - \cancel{\int_0^{ct} (r_0 J_\phi)(0) \, dr_0}\right] \sin\theta_0 \, d\phi_0 \, d\theta_0 \Bigg\}
\end{aligned}$$

But the components of $\mathbf{J}$ are all zero at $t = 0$.

$$\frac{\partial \mathbf{E}_p}{\partial t}(x,y,z,0) = c^2[\nabla \times \mathbf{B}_p(x,y,z,0) - \mu_0 \mathbf{J}(x,y,z,0)] \quad \rightarrow \quad \mathbf{0} = -c^2 \mu_0 \mathbf{J}(x,y,z,0) \quad \rightarrow \quad \mathbf{J}(x,y,z,0) = \mathbf{0}$$

That is, $J_\theta(ct,\phi_0,\theta_0,0) = 0$ and $J_\phi(ct,\phi_0,\theta_0,0) = 0$.

$$\begin{aligned}
\mathbf{B}_p(x,y,z,t) &= \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 \\
&\quad - \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \hat{\boldsymbol{\theta}}_0 \, J_\phi \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0 + \frac{\mu_0}{4\pi}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \hat{\boldsymbol{\phi}}_0 \, J_\theta \sin\theta_0 \, dr_0 \, d\phi_0 \, d\theta_0
\end{aligned}$$

Write the last two integrals in terms of the cross product of position and $\mathbf{J}$.

$$\begin{aligned}
\mathbf{B}_p(x,y,z,t) &= \frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right]r_0\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&\quad+\frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[-\hat{\boldsymbol{\theta}}_0\,J_\phi\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+\hat{\boldsymbol{\phi}}_0\,J_\theta\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right]\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&= \frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right]r_0\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&\quad+\frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[(\hat{\mathbf{r}}_0\times\hat{\mathbf{r}}_0)J_r\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+(\hat{\mathbf{r}}_0\times\hat{\boldsymbol{\phi}}_0)J_\phi\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right. \\
&\qquad\left.+(\hat{\mathbf{r}}_0\times\hat{\boldsymbol{\theta}}_0)J_\theta\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right]\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&= \frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right]r_0\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&\quad+\frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\hat{\mathbf{r}}_0\times\mathbf{J}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&= \frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{r_0}\hat{\mathbf{r}}_0\times\mathbf{J}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+\frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right]r_0\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&= \frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{r_0^2}\mathbf{r}_0\times\mathbf{J}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+\frac{1}{cr_0}\mathbf{r}_0\times\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right]r_0\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&= \frac{\mu_0}{4\pi}\int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{r_0^3}\mathbf{r}_0\times\mathbf{J}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)+\frac{1}{cr_0^2}\mathbf{r}_0\times\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)\right]r_0^2\sin\theta_0\,dr_0\,d\phi_0\,d\theta_0 \\
&= \frac{\mu_0}{4\pi}\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\le c^2t^2}}\left(\frac{1}{r_0^3}\mathbf{r}_0\times\mathbf{J}+\frac{1}{cr_0^2}\mathbf{r}_0\times\frac{\partial\mathbf{J}}{\partial t}\right)\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)dV_0 \\
&= \frac{\mu_0}{4\pi}\iiint\limits_{r_0\le ct}\left(\frac{1}{r_0^3}\mathbf{r}_0\times\mathbf{J}+\frac{1}{cr_0^2}\mathbf{r}_0\times\frac{\partial\mathbf{J}}{\partial t}\right)\left(r_0,\phi_0,\theta_0,t-\frac{r_0}{c}\right)dV_0
\end{aligned}$$

In Cartesian coordinates ($dV_0 = dx_0\,dy_0\,dz_0$) this is written as

$$\begin{aligned}
\mathbf{B}_p(x,y,z,t) = \frac{\mu_0}{4\pi}\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\le c^2t^2}}&\left\{\frac{(x_0-x)\hat{\mathbf{x}}+(y_0-y)\hat{\mathbf{y}}+(z_0-z)\hat{\mathbf{z}}}{\sqrt{[(x_0-x)^2+(y_0-y)^2+(z_0-z)^2]^3}}\right. \\
&\quad\times\mathbf{J}\left(x_0,y_0,z_0,t-\frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right) \\
&+\frac{1}{c}\frac{(x_0-x)\hat{\mathbf{x}}+(y_0-y)\hat{\mathbf{y}}+(z_0-z)\hat{\mathbf{z}}}{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2} \\
&\quad\left.\times\frac{\partial\mathbf{J}}{\partial t}\left(x_0,y_0,z_0,t-\frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right)\right\}dV_0.
\end{aligned}$$

In conclusion, provided that ρ and $\mathbf{J}$ obey the continuity equation, the electric and magnetic fields in a vacuum with no boundaries are

$$\mathbf{E} = \frac{1}{4\pi c^2 t} \iint\limits_{r_0 = ct} \left(\frac{1}{t}\mathbf{E}_0 + c\frac{\partial \mathbf{E}_0}{\partial r_0} + c^2 \nabla \times \mathbf{B}_0 \right) dS_0 - \frac{1}{4\pi\epsilon_0} \iiint\limits_{r_0 \le ct} \left(\frac{\mathbf{r}_0}{r_0^3}\rho + \frac{\mathbf{r}_0}{c r_0^2}\frac{\partial \rho}{\partial t} + \frac{1}{c^2 r_0}\frac{\partial \mathbf{J}}{\partial t} \right) \left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c} \right) dV_0$$

$$\mathbf{B} = \frac{1}{4\pi c^2 t} \iint\limits_{r_0 = ct} \left(\frac{1}{t}\mathbf{B}_0 + c\frac{\partial \mathbf{B}_0}{\partial r_0} - \nabla \times \mathbf{E}_0 \right) dS_0 + \frac{\mu_0}{4\pi} \iiint\limits_{r_0 \le ct} \left(\frac{1}{r_0^3}\mathbf{r}_0 \times \mathbf{J} + \frac{1}{c r_0^2}\mathbf{r}_0 \times \frac{\partial \mathbf{J}}{\partial t} \right) \left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c} \right) dV_0$$

since $\mathbf{E} = \mathbf{E}_c + \mathbf{E}_p$ and $\mathbf{B} = \mathbf{B}_c + \mathbf{B}_p$. The volume integrals are over a solid ball in $x_0 y_0 z_0$-space centered at (x, y, z) with radius ct, and the surface integrals are over this ball's surface. They can be evaluated if the inhomogeneous terms are simple enough; see Strauss 9.2.6 and Strauss 9.2.7 and Strauss 9.3.7 for examples. Observe that $\mathbf{E}$ and $\mathbf{B}$ can be nonzero even in the absence of charges and currents ($\rho = 0$ and $\mathbf{J} = \mathbf{0}$). If instead the initial fields are zero, $\mathbf{E}_0 = \mathbf{0}$ and $\mathbf{B}_0 = \mathbf{0}$, then these formulas reduce to

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0} \iiint\limits_{r_0 \le ct} \left(\frac{\mathbf{r}_0}{r_0^3}\rho + \frac{\mathbf{r}_0}{c r_0^2}\frac{\partial \rho}{\partial t} + \frac{1}{c^2 r_0}\frac{\partial \mathbf{J}}{\partial t} \right) \left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c} \right) dV_0$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \iiint\limits_{r_0 \le ct} \left(\frac{1}{r_0^3}\mathbf{r}_0 \times \mathbf{J} + \frac{1}{c r_0^2}\mathbf{r}_0 \times \frac{\partial \mathbf{J}}{\partial t} \right) \left(r_0, \phi_0, \theta_0, t - \frac{r_0}{c} \right) dV_0,$$

which are the fabled Jefimenko equations. Under electrostatic and magnetostatic conditions, the time derivatives and time dependences vanish ($t \to \infty$), resulting in Coulomb's law and the Biot-Savart law.

Coulomb's Law: $$\mathbf{E} = -\frac{1}{4\pi\epsilon_0} \iiint\limits_{\text{all space}} \frac{\mathbf{r}_0}{r_0^3}\rho \, dV_0 = \frac{1}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(x - x_0)\hat{\mathbf{x}} + (y - y_0)\hat{\mathbf{y}} + (z - z_0)\hat{\mathbf{z}}}{\sqrt{[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2]^3}} \rho(x_0, y_0, z_0) \, dx_0 \, dy_0 \, dz_0$$

Biot-Savart Law: $$\mathbf{B} = \frac{\mu_0}{4\pi} \iiint\limits_{\text{all space}} \frac{1}{r_0^3}\mathbf{r}_0 \times \mathbf{J} \, dV_0$$

$$= \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{[(z_0 - z)J_y - (y_0 - y)J_z]\hat{\mathbf{x}} + [(x_0 - x)J_z - (z_0 - z)J_x]\hat{\mathbf{y}} + [(y_0 - y)J_x - (x_0 - x)J_y]\hat{\mathbf{z}}}{\sqrt{[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2]^3}} \, dx_0 \, dy_0 \, dz_0$$

Appendix A - Kirchhoff and Poisson's Formula

Here the aim is to solve the homogeneous three-dimensional wave equation with two prescribed initial conditions.

$$\begin{aligned}
&u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y, z < \infty,\ t > 0 \\
&u(x, y, z, 0) = \alpha(x, y, z) \\
&u_t(x, y, z, 0) = \beta(x, y, z)
\end{aligned}$$

Start by looking at the one-dimensional wave equation.

$$u_{tt} - c^2 u_{xx} = 0$$

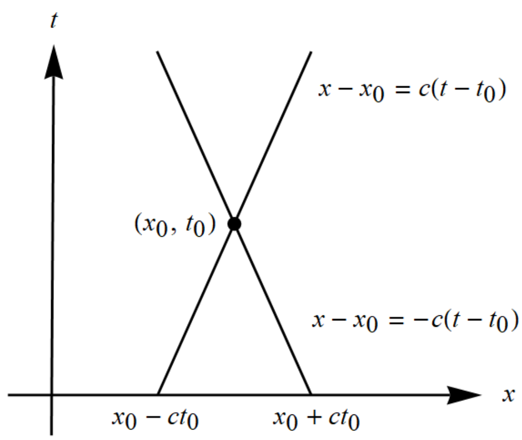
Comparing this with the general form of a linear second-order PDE, $Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G$, we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The discriminant, $B^2 - 4AC = 4c^2$, is greater than 0, which means the PDE is hyperbolic. That means two real and distinct families of characteristic curves exist in the xt-plane.

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dx}{dt} &= \frac{1}{2}\left(\pm\sqrt{0 + 4c^2}\right) \\
\frac{dx}{dt} &= c \quad \text{or} \quad \frac{dx}{dt} = -c
\end{aligned}$$

Integrate both sides of each ODE to get x.

$$x = ct + \xi \quad \text{or} \quad x = -ct + \eta$$

ξ and η are the characteristic coordinates (basically the names of the members in the respective families). Suppose we want to evaluate u at (x_0, t_0) and that ξ_0 and η_0 are the coordinates for the lines going through this point.



The equations for the lines going through this point are given by the point-slope formula.

$$x - x_0 = \pm c(t - t_0)$$

This can be generalized to three dimensions by rotating the line with slope $+c$ about the vertical axis going through (x_0, t_0).

$$|\mathbf{x} - \mathbf{x}_0| = c|t - t_0|.$$

This is now a characteristic surface (known as a light cone because it has slope c) in space-time that goes through the point (x_0, y_0, z_0, t_0). Consider the backward light cone of the point $(0, 0, 0, t_0)$.

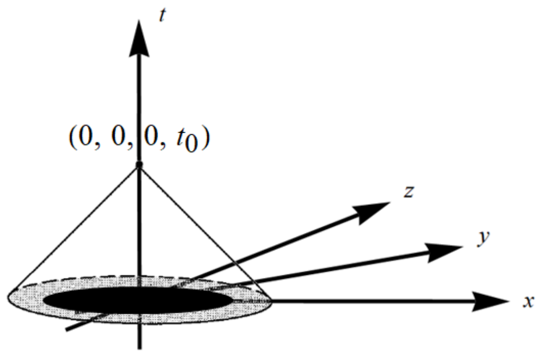


Figure 1: The black hyperdisk shown lies in the xyz-plane, has center $(0, 0, 0)$ and radius r, and represents a solid ball in xyz-space. Note that the shaded hyperdisk it lies within has radius ct_0, where t_0 is a particular time we want to evaluate u at.

Integrate both sides of the three-dimensional wave equation over this black hyperdisk.

$$\iiint_V u_{tt}\, dV = \iiint_V c^2 \nabla^2 u\, dV$$

$$\iiint_V u_{tt}\, dV = c^2 \iiint_V \nabla \cdot \nabla u\, dV$$

Apply the divergence theorem to the volume integral on the right side to turn it into a surface integral over the solid ball's boundary.

$$\iiint_V u_{tt}\, dV = c^2 \oiint_S \nabla u \cdot \hat{\mathbf{n}}\, dS$$

The unit vector normal to the boundary is the radial unit vector: $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. $\nabla u \cdot \hat{\mathbf{r}}$ can be interpreted as the directional derivative in the radial direction, that is, $\partial u/\partial r$.

$$\iiint_V \frac{\partial^2 u}{\partial t^2}\, dV = c^2 \oiint_S \frac{\partial u}{\partial r}\, dS$$

Write out the volume and surface integrals explicitly by using spherical coordinates (r, ϕ, θ). Here θ denotes the angle from the polar axis.

$$\int_0^\pi \int_0^{2\pi} \int_0^r \frac{\partial^2 u}{\partial t^2}(r'^2 \sin\theta\, dr'\, d\phi\, d\theta) = c^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r}(r^2 \sin\theta\, d\phi\, d\theta)$$

$$\int_0^r r'^2 \int_0^\pi \int_0^{2\pi} \frac{\partial^2 u}{\partial t^2} \sin\theta\, d\phi\, d\theta\, dr' = c^2 r^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r} \sin\theta\, d\phi\, d\theta$$

Bring the derivatives out in front.

$$\int_0^r r'^2 \frac{\partial^2}{\partial t^2}\left[\int_0^\pi \int_0^{2\pi} u(r', \phi, \theta, t) \sin\theta \, d\phi \, d\theta\right] dr' = c^2 r^2 \frac{\partial}{\partial r}\left[\int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin\theta \, d\phi \, d\theta\right]$$

Let

$$v(r, t) = \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin\theta \, d\phi \, d\theta$$

so that the previous equation becomes

$$\int_0^r r'^2 \frac{\partial^2 v}{\partial t^2} \, dr' = c^2 r^2 \frac{\partial v}{\partial r}.$$

Differentiate both sides with respect to r to eliminate the integral on the left side.

$$r^2 \frac{\partial^2 v}{\partial t^2} = c^2 \left(2r \frac{\partial v}{\partial r} + r^2 \frac{\partial^2 v}{\partial r^2}\right)$$

Divide both sides by r.

$$r \frac{\partial^2 v}{\partial t^2} = c^2 \left(2 \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2}\right)$$

Now make the change of variables $w(r, t) = rv(r, t)$. Write the new derivatives in terms of the old ones by differentiating this substitution.

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= r \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial w}{\partial r} &= v + r \frac{\partial v}{\partial r} \\ \frac{\partial^2 w}{\partial r^2} &= 2 \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2} \end{aligned}$$

The transformed PDE is the one-dimensional wave equation on a semi-infinite interval

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial r^2}, \quad 0 < r < \infty, \ t > 0$$

subject to the Dirichlet boundary condition $w(0, t) = 0$ and the following initial conditions.

$$w(r, 0) = r \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, 0) \sin\theta \, d\phi \, d\theta = r \int_0^\pi \int_0^{2\pi} \alpha(r, \phi, \theta) \sin\theta \, d\phi \, d\theta$$

$$\frac{\partial w}{\partial t}(r, 0) = r \int_0^\pi \int_0^{2\pi} u_t(r, \phi, \theta, 0) \sin\theta \, d\phi \, d\theta = r \int_0^\pi \int_0^{2\pi} \beta(r, \phi, \theta) \sin\theta \, d\phi \, d\theta$$

The method of reflection can be applied to solve the one-dimensional wave equation. Consider the corresponding problem over the whole line,

$$W_{tt} = c^2 W_{rr}, \quad -\infty < r < \infty,\ t > 0$$
$$W(r,0) = A_{\text{odd}}(r), \quad W_t(r,0) = B_{\text{odd}}(r),$$

where the odd extensions of the initial conditions for w, $A_{\text{odd}}(r)$ and $B_{\text{odd}}(r)$, are used in order to satisfy the Dirichlet boundary condition.

$$A_{\text{odd}}(r) = \begin{cases} r\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(r,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r > 0 \\ -(-r)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(-r,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r < 0 \end{cases}$$

$$B_{\text{odd}}(r) = \begin{cases} r\displaystyle\int_0^\pi \int_0^{2\pi} \beta(r,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r > 0 \\ -(-r)\displaystyle\int_0^\pi \int_0^{2\pi} \beta(-r,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r < 0 \end{cases}$$

The solution for W is given by d'Alembert's formula (see Appendix B).

$$W(r,t) = \frac{1}{2}[A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} B_{\text{odd}}(s)\, ds$$

The solution for w is then just the restriction of W to $r > 0$.

$$w(r,t) = \frac{1}{2}[A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} B_{\text{odd}}(s)\, ds, \quad r > 0$$

Our task now is to write this formula in terms of the given functions, α and β. Note that

$$A_{\text{odd}}(r+ct) = \begin{cases} (r+ct)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(r+ct,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r+ct > 0 \\ -(-r-ct)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(-r-ct,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r+ct < 0 \end{cases}$$

and

$$A_{\text{odd}}(r-ct) = \begin{cases} (r-ct)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(r-ct,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r-ct > 0 \\ -(-r+ct)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(-r+ct,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r-ct < 0 \end{cases},$$

so for every region in the rt-quarter-plane, we have to test whether $r - ct$ and $r + ct$ are greater than or less than zero.

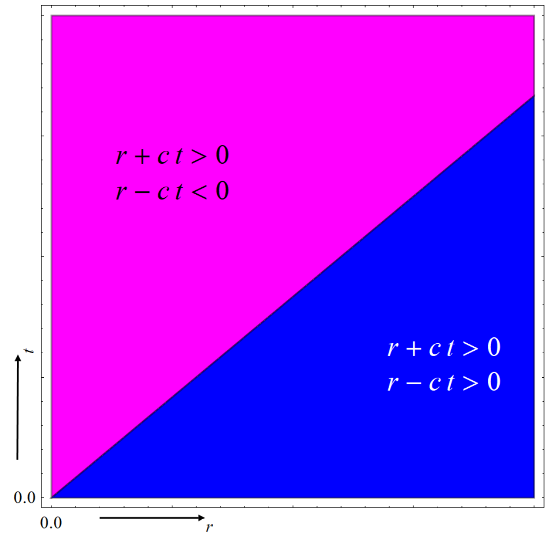


Figure 2: This figure illustrates the regions in the rt-quarter-plane that come about from using the odd extensions of A and B. The solution for w has to be determined in each one. The characteristic curve $r - ct = 0$ is the line that separates the regions.

The Magenta Region

In the magenta region $r + ct > 0$ and $r - ct < 0$, so the solution for w is

$$\begin{aligned}
w(r,t) &= \frac{1}{2}[A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} B_{\text{odd}}(s)\,ds \\
&= \frac{1}{2}\left[(r+ct)\int_0^\pi\int_0^{2\pi}\alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta - (-r+ct)\int_0^\pi\int_0^{2\pi}\alpha(-r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\
&\qquad + \frac{1}{2c}\left[\int_{r-ct}^0 -(-s)\int_0^\pi\int_0^{2\pi}\beta(-s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds + \int_0^{r+ct} s\int_0^\pi\int_0^{2\pi}\beta(s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds\right].
\end{aligned}$$

Substitute $p = -s$ in the third integral and substitute $p = s$ in the fourth integral.

$$\begin{aligned}
&= \frac{1}{2}\left[\int_0^\pi\int_0^{2\pi}(r+ct)\alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta - \int_0^\pi\int_0^{2\pi}(-r+ct)\alpha(-r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\
&\qquad + \frac{1}{2c}\left[\int_{-r+ct}^0 p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp + \int_0^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] \\
&= \frac{1}{2}\int_0^\pi\int_0^{2\pi}[(r+ct)\alpha(r+ct,\phi,\theta) - (-r+ct)\alpha(-r+ct,\phi,\theta)]\sin\theta\,d\phi\,d\theta \\
&\qquad + \frac{1}{2c}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp
\end{aligned}$$

The expression in square brackets can be written as the derivative of an integral by the Leibnitz rule.

$$\begin{aligned}
&= \frac{1}{2}\int_0^\pi\int_0^{2\pi}\left[\frac{1}{c}\frac{\partial}{\partial t}\int_{-r+ct}^{r+ct} p\alpha(p,\phi,\theta)\,dp\right]\sin\theta\,d\phi\,d\theta + \frac{1}{2c}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp \\
&= \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp
\end{aligned}$$

The Blue Region

In the blue region $r + ct > 0$ and $r - ct > 0$, so the solution for w is

$$
\begin{aligned}
w(r,t) &= \frac{1}{2}[A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} B_{\text{odd}}(s)\,ds \\
&= \frac{1}{2}\left[(r+ct)\int_0^\pi \int_0^{2\pi} \alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta + (r-ct)\int_0^\pi \int_0^{2\pi} \alpha(r-ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\
&\quad + \frac{1}{2c}\int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds \\
&= \frac{1}{2}\left[\int_0^\pi \int_0^{2\pi} (r+ct)\alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta + \int_0^\pi \int_0^{2\pi} (r-ct)\alpha(r-ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\
&\quad + \frac{1}{2c}\int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds \\
&= \frac{1}{2}\int_0^\pi \int_0^{2\pi} [(r+ct)\alpha(r+ct,\phi,\theta) + (r-ct)\alpha(r-ct,\phi,\theta)]\sin\theta\,d\phi\,d\theta \\
&\quad + \frac{1}{2c}\int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds.
\end{aligned}
$$

The expression in square brackets can be written as the derivative of an integral by the Leibnitz rule. Also, let $p = s$ in the second integral.

$$
\begin{aligned}
&= \frac{1}{2}\int_0^\pi \int_0^{2\pi} \left[\frac{1}{c}\frac{\partial}{\partial t}\int_{r-ct}^{r+ct} p\alpha(p,\phi,\theta)\,dp\right]\sin\theta\,d\phi\,d\theta + \frac{1}{2c}\int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp \\
&= \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp
\end{aligned}
$$

Consequently,

$$w(r,t) = \begin{cases} \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{-r+ct}^{r+ct} p \int_0^{\pi}\int_0^{2\pi} \alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\int_{-r+ct}^{r+ct} p \int_0^{\pi}\int_0^{2\pi} \beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp & \text{if } r-ct<0 \\ \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{r-ct}^{r+ct} p \int_0^{\pi}\int_0^{2\pi} \alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} p \int_0^{\pi}\int_0^{2\pi} \beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp & \text{if } r-ct>0 \end{cases}.$$

Since $v = w/r$, we have

$$\int_0^{\pi}\int_0^{2\pi} u(r,\phi,\theta,t)\sin\theta\,d\phi\,d\theta = \frac{w(r,t)}{r}.$$

In order to calculate the value of u at the origin, take the limit of both sides as $r \to 0$.

$$\lim_{r\to 0}\int_0^{\pi}\int_0^{2\pi} u(r,\phi,\theta,t)\sin\theta\,d\phi\,d\theta = \lim_{r\to 0}\frac{w(r,t)}{r}$$

The value of u at $r = 0$ is $u(x = 0, y = 0, z = 0, t)$. It does not depend on ϕ and θ, so it can be pulled in front of the integral.

$$u(0,0,0,t)\int_0^{\pi}\int_0^{2\pi} \sin\theta\,d\phi\,d\theta = \lim_{r\to 0}\frac{w(r,t)-w(0,t)}{r-0}$$

Evaluate the integral on the left side. The limit on the right side is how the first derivative of w with respect to r at $r = 0$ is defined. The formula for w in the magenta region applies for this value of r. Use the Leibnitz rule to differentiate the integrals with respect to r.

$$\begin{aligned} 4\pi u(0,0,0,t) &= \left.\frac{\partial w}{\partial r}\right|_{r=0} \\ &= \left.\left\{\frac{\partial}{\partial t}\left[\frac{1}{2c}\frac{\partial}{\partial r}\int_{-r+ct}^{r+ct} p\int_0^{\pi}\int_0^{2\pi}\alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\frac{\partial}{\partial r}\int_{-r+ct}^{r+ct} p\int_0^{\pi}\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right\}\right|_{r=0} \\ &= \left\{\frac{1}{2c}\frac{\partial}{\partial t}\left[(r+ct)\int_0^{\pi}\int_0^{2\pi}\alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta - (-r+ct)(-1)\int_0^{\pi}\int_0^{2\pi}\alpha(-r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right]\right. \\ &\qquad \left.\left. + \frac{1}{2c}\left[(r+ct)\int_0^{\pi}\int_0^{2\pi}\beta(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta - (-r+ct)(-1)\int_0^{\pi}\int_0^{2\pi}\beta(-r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right]\right\}\right|_{r=0} \\ &= \frac{1}{2c}\frac{\partial}{\partial t}\left[ct\int_0^{\pi}\int_0^{2\pi}\alpha(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta + ct\int_0^{\pi}\int_0^{2\pi}\alpha(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\ &\qquad + \frac{1}{2c}\left[ct\int_0^{\pi}\int_0^{2\pi}\beta(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta + ct\int_0^{\pi}\int_0^{2\pi}\beta(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \end{aligned}$$

As a result,

$$\begin{aligned} 4\pi u(0,0,0,t) &= \frac{\partial}{\partial t}\left[t\int_0^{\pi}\int_0^{2\pi}\alpha(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] + t\int_0^{\pi}\int_0^{2\pi}\beta(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta \\ &= \frac{\partial}{\partial t}\left[\frac{1}{c^2t}\int_0^{\pi}\int_0^{2\pi}\alpha(ct,\phi,\theta)(c^2t^2\sin\theta\,d\phi\,d\theta)\right] + \frac{1}{c^2t}\int_0^{\pi}\int_0^{2\pi}\beta(ct,\phi,\theta)(c^2t^2\sin\theta\,d\phi\,d\theta). \end{aligned}$$

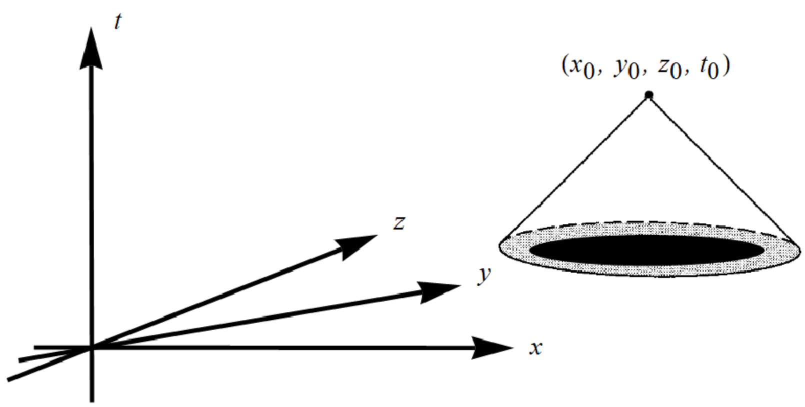
Divide both sides by 4π.

$$u(0,0,0,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2t}\int_0^{\pi}\int_0^{2\pi}\alpha(ct,\phi,\theta)(c^2t^2\sin\theta\,d\phi\,d\theta)\right] + \frac{1}{4\pi c^2t}\int_0^{\pi}\int_0^{2\pi}\beta(ct,\phi,\theta)(c^2t^2\sin\theta\,d\phi\,d\theta)$$

At a particular time $t = t_0$ these double integrals are surface integrals over a sphere of radius ct_0 centered at $(0,0,0)$.

$$u(0,0,0,t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2t_0}\iint\limits_{x^2+y^2+z^2=c^2t_0^2}\alpha(x,y,z)\,dS\right] + \frac{1}{4\pi c^2t_0}\iint\limits_{x^2+y^2+z^2=c^2t_0^2}\beta(x,y,z)\,dS$$

This is the solution of the wave equation at the origin of the xyz-plane. Now we aim to find the solution at a particular point in space-time (x_0, y_0, z_0, t_0).



The wave equation is invariant to translations in space, so if $u(x,y,z,t)$ is a solution to the wave equation

$$\begin{aligned} &u_{tt} = c^2\nabla^2 u, \quad -\infty < x,y,z < \infty,\ t > 0 \\ &u(x,y,z,0) = \alpha(x,y,z) \\ &u_t(x,y,z,0) = \beta(x,y,z), \end{aligned}$$

then $u(x+x_0, y+y_0, z+z_0, t)$ is also a solution to the wave equation, albeit with different initial conditions.

$$\begin{aligned} &u_{tt} = c^2\nabla^2 u, \quad -\infty < x,y,z < \infty,\ t > 0 \\ &u(x+x_0, y+y_0, z+z_0, 0) = \alpha(x+x_0, y+y_0, z+z_0) \\ &u_t(x+x_0, y+y_0, z+z_0, 0) = \beta(x+x_0, y+y_0, z+z_0) \end{aligned}$$

Since the solution to the wave equation is unique, $u(x_0, y_0, z_0, t_0)$ has the same form as $u(0, 0, 0, t_0)$.

$$u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \iint_{x^2+y^2+z^2=c^2t_0^2} \alpha(x+x_0, y+y_0, z+z_0)\, dS\right] + \frac{1}{4\pi c^2 t_0} \iint_{x^2+y^2+z^2=c^2t_0^2} \beta(x+x_0, y+y_0, z+z_0)\, dS$$

Let $k = x + x_0$, $l = y + y_0$, and $m = z + z_0$. Then $x = k - x_0$, $y = l - y_0$, and $z = m - z_0$.

$$u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \iint_{\substack{(k-x_0)^2+(l-y_0)^2\\+(m-z_0)^2=c^2t_0^2}} \alpha(k, l, m)\, dS\right] + \frac{1}{4\pi c^2 t_0} \iint_{\substack{(k-x_0)^2+(l-y_0)^2\\+(m-z_0)^2=c^2t_0^2}} \beta(k, l, m)\, dS$$

k, l, and m are dummy integration variables, so they can be replaced with x, y, and z.

$$u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \iint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=c^2t_0^2}} \alpha(x, y, z)\, dS\right] + \frac{1}{4\pi c^2 t_0} \iint_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=c^2t_0^2}} \beta(x, y, z)\, dS$$

Finally, switch the roles of x, y, z, and t with those of x_0, y_0, z_0, and t_0, respectively, to obtain Kirchhoff and Poisson's formula.

$$\boxed{u(x, y, z, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \iint_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \alpha(x_0, y_0, z_0)\, dS_0\right] + \frac{1}{4\pi c^2 t} \iint_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \beta(x_0, y_0, z_0)\, dS_0}$$

These double integrals are surface integrals over a sphere of radius ct centered at (x, y, z).

Appendix B - d'Alembert's Formula

Here the aim is to solve the homogeneous one-dimensional wave equation with two prescribed initial conditions.

$$\begin{aligned}
&u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty,\ t > 0 \\
&u(x,0) = \alpha(x) \\
&u_t(x,0) = \beta(x)
\end{aligned}$$

Since $-\infty < x < \infty$, the method of operator factorization can be used to solve for u. Bring both terms in the PDE to the left side and factor the operator.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0$$

If we let

$$v = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u,$$

then the PDE becomes

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v = 0.$$

To summarize, the method of operator factorization reduces the wave equation to a system of first-order PDEs, which can both be solved with the method of characteristics.

$$\begin{cases} \dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x} = v(x,t) \\[2ex] \dfrac{\partial v}{\partial t} + c\dfrac{\partial v}{\partial x} = 0 \end{cases}$$

The differential of a two-dimensional function $h(x,t)$ is defined by

$$dh = \frac{\partial h}{\partial t}\,dt + \frac{\partial h}{\partial x}\,dx.$$

Dividing both sides by dt yields the fundamental relationship between the total derivative of h and its partial derivatives.

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\frac{dx}{dt}$$

Along the curves in the xt-plane defined by

$$\frac{dx}{dt} = c, \quad x(\xi, 0) = \xi, \tag{1}$$

where ξ is a characteristic coordinate, the PDE for v becomes an ODE.

$$\frac{dv}{dt} = 0 \tag{2}$$

Integrate both sides of the ODE in equation (1) with respect to t.

$$x = ct + \xi \quad \rightarrow \quad \xi = x - ct$$

Now integrate both sides of equation (2) with respect to t.

$$v(\xi, t) = f(\xi)$$

Here f is an arbitrary function of the characteristic coordinate. Now that v is solved for, write ξ in terms of x and t.

$$v(x, t) = f(x - ct)$$

The PDE for u then becomes

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = f(x - ct).$$

Use the method of characteristics again. Along the curves in the xt-plane defined by

$$\frac{dx}{dt} = -c, \quad x(\eta, 0) = \eta, \tag{3}$$

where η is another characteristic coordinate, the PDE for u becomes an ODE.

$$\frac{du}{dt} = f(x - ct) \tag{4}$$

Integrate both sides of the ODE in equation (3) with respect to t.

$$x = -ct + \eta \quad \rightarrow \quad \eta = x + ct$$

With this formula for x, equation (4) becomes

$$\frac{du}{dt} = f(\eta - 2ct).$$

Integrate both sides with respect to t.

$$u(\eta, t) = F(\eta - 2ct) + G(\eta)$$

Here F and G are new arbitrary functions. Now that u is solved for, replace η with $x + ct$.

$$u(x, t) = F(x - ct) + G(x + ct)$$

This is the general solution of the wave equation. Take the derivative of it with respect to t and use the chain rule.

$$\frac{\partial u}{\partial t} = -cF'(x - ct) + cG'(x + ct)$$

Apply the initial conditions now to determine F and G.

$$u(x, 0) = F(x) + G(x) = \alpha(x)$$
$$\frac{\partial u}{\partial t}(x, 0) = -cF'(x) + cG'(x) = \beta(x)$$

Differentiate both sides of the first equation and multiply both its sides by c.

$$cF'(x) + cG'(x) = c\alpha'(x) \tag{5}$$
$$-cF'(x) + cG'(x) = \beta(x) \tag{6}$$

Add the respective sides of equations (5) and (6) to eliminate $F'(x)$.

$$2cG'(x) = c\alpha'(x) + \beta(x)$$

Solve for $G(x)$.

$$G'(x) = \frac{1}{2}\alpha'(x) + \frac{1}{2c}\beta(x)$$

$$G(x) = \frac{1}{2}\alpha(x) + \frac{1}{2c}\int^x \beta(s)\,ds + C_1$$

What we solved for is actually $G(w)$, where w is any expression we wish. That means

$$G(x+ct) = \frac{1}{2}\alpha(x+ct) + \frac{1}{2c}\int^{x+ct} \beta(s)\,ds + C_1.$$

Subtract the respective sides of equations (5) and (6) to eliminate $G'(x)$.

$$2cF'(x) = c\alpha'(x) - \beta(x)$$

Solve for $F(x)$.

$$F'(x) = \frac{1}{2}\alpha'(x) - \frac{1}{2c}\beta(x)$$

$$F(x) = \frac{1}{2}\alpha(x) - \frac{1}{2c}\int^x \beta(s)\,ds + C_2$$

What we solved for is actually $F(w)$, where w is any expression we wish. That means

$$F(x-ct) = \frac{1}{2}\alpha(x-ct) - \frac{1}{2c}\int^{x-ct} \beta(s)\,ds + C_1.$$

So then

$$\begin{aligned}
u(x,t) &= F(x-ct) + G(x+ct) \\
&= \frac{1}{2}[\alpha(x-ct) + \alpha(x+ct)] - \frac{1}{2c}\int^{x-ct} \beta(s)\,ds + \frac{1}{2c}\int^{x+ct} \beta(s)\,ds + C_1 + C_2 \\
&= \frac{1}{2}[\alpha(x-ct) + \alpha(x+ct)] + \frac{1}{2c}\int_{x-ct} \beta(s)\,ds + \frac{1}{2c}\int^{x+ct} \beta(s)\,ds + C_1 + C_2 \\
&= \frac{1}{2}[\alpha(x-ct) + \alpha(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta(s)\,ds + C_1 + C_2
\end{aligned}$$

Therefore, setting the unnecessary integration constants to zero, we obtain d'Alembert's formula.

$$\boxed{u(x,t) = \frac{1}{2}[\alpha(x-ct) + \alpha(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta(s)\,ds}$$

Appendix C - The Leibnitz Rule

The Leibnitz rule is an extension of the fundamental theorem of calculus.

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(x,t)\,dt = \int_{g(x)}^{h(x)} \frac{\partial f}{\partial x}(x,t)\,dt + \frac{dh}{dx}f(x,h(x)) - \frac{dg}{dx}f(x,g(x))$$