Vector Identity 1

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

Proof

Let $\mathbf{A}=\mathbf{a}$, $\mathbf{B}=\mathbf{b}$, and $\mathbf{C}=\mathbf{c}$.

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\cdot\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j B_j\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k C_k\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\cdot\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)B_jC_k\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\cdot\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{ljk}B_jC_k\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_l)\varepsilon_{ljk}A_iB_jC_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\varepsilon_{ljk}A_iB_jC_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}A_iB_jC_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{kij}A_iB_jC_k \qquad (1)\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{kl}\varepsilon_{lij}A_iB_jC_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_l)\varepsilon_{lij}A_iB_jC_k \\
&= \left(\sum_{k=1}^{3}\boldsymbol{\delta}_k C_k\right)\cdot\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{lij}A_iB_j\right) \\
&= \left(\sum_{k=1}^{3}\boldsymbol{\delta}_k C_k\right)\cdot\left[\sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)A_iB_j\right] \\
&= \left(\sum_{k=1}^{3}\boldsymbol{\delta}_k C_k\right)\cdot\left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i A_i\right)\times\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j B_j\right)\right] \\
&= \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})
\end{aligned}$$

To get the last identity, return to equation (1).

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{kij}A_iB_jC_k \qquad (1)\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{jki}A_iB_jC_k\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jl}\varepsilon_{lki}A_iB_jC_k\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_l)\varepsilon_{lki}A_iB_jC_k\\
&= \left(\sum_{j=1}^{3}\boldsymbol{\delta}_jB_j\right)\cdot\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{lki}A_iC_k\right)\\
&= \left(\sum_{j=1}^{3}\boldsymbol{\delta}_jB_j\right)\cdot\left[\sum_{i=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_k\times\boldsymbol{\delta}_i)A_iC_k\right]\\
&= \left(\sum_{j=1}^{3}\boldsymbol{\delta}_jB_j\right)\cdot\left[\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kC_k\right)\times\left(\sum_{i=1}^{3}\boldsymbol{\delta}_iA_i\right)\right]\\
&= \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})
\end{aligned}$$

Therefore,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}).$$