

Vector Identity 10

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

Proof

Let $\mathbf{A} = \mathbf{a}$ and $\mathbf{B} = \mathbf{b}$.

$$\begin{aligned} \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \cdot \left[\left(\sum_{j=1}^3 \delta_j A_j \right) \times \left(\sum_{k=1}^3 \delta_k B_k \right) \right] \\ &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \cdot \left[\sum_{j=1}^3 \sum_{k=1}^3 (\delta_j \times \delta_k) A_j B_k \right] \\ &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \cdot \left(\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_l \varepsilon_{jkl} A_j B_k \right) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_i \cdot \delta_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_{il} \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{jki} \frac{\partial}{\partial x_i} A_j B_k \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \frac{\partial}{\partial x_i} A_j B_k \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \left(\frac{\partial A_j}{\partial x_i} B_k + A_j \frac{\partial B_k}{\partial x_i} \right) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_k + \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} A_j \frac{\partial B_k}{\partial x_i} \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_k + \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 (-\varepsilon_{ikj}) A_j \frac{\partial B_k}{\partial x_i} \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_{lk} \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_l - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_{lj} \varepsilon_{ikj} A_l \frac{\partial B_k}{\partial x_i} \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_l \cdot \delta_k) \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} B_l - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_l \cdot \delta_j) \varepsilon_{ikj} A_l \frac{\partial B_k}{\partial x_i} \\ &= \left(\sum_{l=1}^3 \delta_l B_l \right) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \varepsilon_{ijk} \frac{\partial A_j}{\partial x_i} \right) - \left(\sum_{l=1}^3 \delta_l A_l \right) \cdot \left(\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \varepsilon_{ikj} \frac{\partial B_k}{\partial x_i} \right) \end{aligned}$$

Continue the simplification.

$$\begin{aligned}\nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \left(\sum_{l=1}^3 \delta_l B_l \right) \cdot \left[\sum_{i=1}^3 \sum_{j=1}^3 (\delta_i \times \delta_j) \frac{\partial A_j}{\partial x_i} \right] - \left(\sum_{l=1}^3 \delta_l A_l \right) \cdot \left[\sum_{i=1}^3 \sum_{k=1}^3 (\delta_i \times \delta_k) \frac{\partial B_k}{\partial x_i} \right] \\ &= \left(\sum_{l=1}^3 \delta_l B_l \right) \cdot \left[\left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left(\sum_{j=1}^3 \delta_j A_j \right) \right] - \left(\sum_{l=1}^3 \delta_l A_l \right) \cdot \left[\left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left(\sum_{k=1}^3 \delta_k B_k \right) \right] \\ &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})\end{aligned}$$