Vector Identity 11

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

Proof

Let $\mathbf{A} = \mathbf{a}$ and $\mathbf{B} = \mathbf{b}$.

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) A_j B_k\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} A_j B_k\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{mil} \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mj}\delta_{ik} \frac{\partial}{\partial x_i} A_j B_k - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mk}\delta_{ij} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \delta_{ik} \frac{\partial}{\partial x_i} A_j B_k - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \delta_{ij} \frac{\partial}{\partial x_i} A_j B_k \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_k} A_j B_k - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_j} A_j B_k \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \left(\frac{\partial A_j}{\partial x_k} B_k + A_j \frac{\partial B_k}{\partial x_k}\right) - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \left(\frac{\partial A_j}{\partial x_j} B_k + A_j \frac{\partial B_k}{\partial x_j}\right) \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \frac{\partial A_j}{\partial x_k} B_k + \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j A_j \frac{\partial B_k}{\partial x_k} - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial A_j}{\partial x_j} B_k - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k A_j \frac{\partial B_k}{\partial x_j} \\
&= \sum_{k=1}^{3} B_k \frac{\partial}{\partial x_k} \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) + \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j A_j\right) \sum_{k=1}^{3} \frac{\partial B_k}{\partial x_k} - \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k\right) \sum_{j=1}^{3} \frac{\partial A_j}{\partial x_j} - \sum_{j=1}^{3} A_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k B_k\right) \\
&= (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B}
\end{aligned}$$