Vector Identity 16

$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a}-\mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$

Proof

The gradient operator in spherical coordinates is given by

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

That means

$$\begin{aligned}\mathbf{a}\cdot\nabla &= (a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\phi\hat{\boldsymbol{\phi}})\cdot\left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\\ &= a_r\frac{\partial}{\partial r} + a_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + a_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi},\end{aligned}$$

and

$$\begin{aligned}(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) &= (\mathbf{a}\cdot\nabla)\hat{\mathbf{r}}f(r)\\ &= \left(a_r\frac{\partial}{\partial r} + a_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + a_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\hat{\mathbf{r}}f(r)\\ &= a_r\frac{\partial}{\partial r}[\hat{\mathbf{r}}f(r)] + a_\theta\frac{1}{r}\frac{\partial}{\partial\theta}[\hat{\mathbf{r}}f(r)] + a_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}[\hat{\mathbf{r}}f(r)]\\ &= a_r\left[\frac{\partial\hat{\mathbf{r}}}{\partial r}f(r) + \hat{\mathbf{r}}\frac{df}{dr}\right] + a_\theta\frac{f(r)}{r}\frac{\partial\hat{\mathbf{r}}}{\partial\theta} + a_\phi\frac{f(r)}{r\sin\theta}\frac{\partial\hat{\mathbf{r}}}{\partial\phi}\\ &= a_r\left[(0)f(r) + \hat{\mathbf{r}}\frac{df}{dr}\right] + a_\theta\frac{f(r)}{r}(\hat{\boldsymbol{\theta}}) + a_\phi\frac{f(r)}{r\sin\theta}(\hat{\boldsymbol{\phi}}\sin\theta)\\ &= \hat{\mathbf{r}}\,a_r\frac{df}{dr} + \hat{\boldsymbol{\theta}}\,a_\theta\frac{f(r)}{r} + \hat{\boldsymbol{\phi}}\,a_\phi\frac{f(r)}{r}\\ &= \hat{\mathbf{r}}\,a_r\frac{df}{dr} + \frac{f(r)}{r}\left(\hat{\boldsymbol{\theta}}\,a_\theta + \hat{\boldsymbol{\phi}}\,a_\phi\right)\\ &= \hat{\mathbf{r}}\,a_r\frac{df}{dr} + \frac{f(r)}{r}\left(\hat{\mathbf{r}}\,a_r + \hat{\boldsymbol{\theta}}\,a_\theta + \hat{\boldsymbol{\phi}}\,a_\phi - \hat{\mathbf{r}}\,a_r\right)\\ &= \hat{\mathbf{r}}\,a_r\frac{df}{dr} + \frac{f(r)}{r}\left(\mathbf{a} - \hat{\mathbf{r}}\,a_r\right)\\ &= \hat{\mathbf{r}}(\mathbf{a}\cdot\hat{\mathbf{r}})\frac{df}{dr} + \frac{f(r)}{r}\left[\mathbf{a} - \hat{\mathbf{r}}(\mathbf{a}\cdot\hat{\mathbf{r}})\right]\\ &= \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{df}{dr} + \frac{f(r)}{r}\left[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\right]\\ &= \frac{f(r)}{r}\left[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\right] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{df}{dr}.\end{aligned}$$