Vector Identity 17

$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \dfrac{1}{i}(\mathbf{x}\times\nabla)$ is the angular momentum operator.

Proof

$$\begin{aligned}
i(\mathbf{L}\times\mathbf{a}) &= i\left\{\left[\frac{1}{i}(\mathbf{x}\times\nabla)\right]\times\mathbf{a}\right\} \\
&= (\mathbf{x}\times\nabla)\times\mathbf{a} \\
&= \left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i x_i\right)\times\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\right]\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k a_k\right) \\
&= \left[\sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)x_i\frac{\partial}{\partial x_j}\right]\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k a_k\right) \\
&= \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{lij}x_i\frac{\partial}{\partial x_j}\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k a_k\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_l\times\boldsymbol{\delta}_k)\varepsilon_{lij}x_i\frac{\partial}{\partial x_j}a_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mlk}\varepsilon_{lij}x_i\frac{\partial}{\partial x_j}a_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{lkm}\varepsilon_{lij}x_i\frac{\partial}{\partial x_j}a_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(\delta_{ki}\delta_{mj}-\delta_{kj}\delta_{mi})x_i\frac{\partial}{\partial x_j}a_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{ki}\delta_{mj}x_i\frac{\partial}{\partial x_j}a_k - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{kj}\delta_{mi}x_i\frac{\partial}{\partial x_j}a_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\delta_{ki}x_i\frac{\partial}{\partial x_j}a_k - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_i\delta_{kj}x_i\frac{\partial}{\partial x_j}a_k \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j x_k\frac{\partial}{\partial x_j}a_k - \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i x_i\frac{\partial}{\partial x_j}a_j
\end{aligned}$$

Note

$$\frac{\partial}{\partial x_j} x_k a_k = x_k \frac{\partial}{\partial x_j} a_k + a_k \frac{\partial}{\partial x_j} x_k$$

and continue the simplification.

$$\begin{aligned}
i(\mathbf{L} \times \mathbf{a}) &= \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_j \left(\frac{\partial}{\partial x_j} x_k a_k - a_k \frac{\partial}{\partial x_j} x_k \right) - \sum_{i=1}^{3} \sum_{j=1}^{3} \boldsymbol{\delta}_i x_i \frac{\partial}{\partial x_j} a_j \\
&= \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j} x_k a_k - \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_j a_k \frac{\partial}{\partial x_j} x_k - \sum_{i=1}^{3} \sum_{j=1}^{3} \boldsymbol{\delta}_i x_i \frac{\partial}{\partial x_j} a_j \\
&= \sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^{3} x_k a_k \right) - \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_j a_k \delta_{jk} - \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i \right) \sum_{j=1}^{3} \frac{\partial}{\partial x_j} a_j \\
&= \sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^{3} x_k a_k \right) - \sum_{j=1}^{3} \boldsymbol{\delta}_j a_j - \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i \right) \sum_{j=1}^{3} \frac{\partial}{\partial x_j} a_j \\
&= \nabla(\mathbf{x} \cdot \mathbf{a}) - \mathbf{a} - \mathbf{x}(\nabla \cdot \mathbf{a})
\end{aligned}$$

Therefore,

$$\nabla(\mathbf{x} \cdot \mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla \cdot \mathbf{a}) + i(\mathbf{L} \times \mathbf{a}),$$

where

$$\mathbf{L} = \frac{1}{i}(\mathbf{x} \times \nabla).$$