Vector Identity 18

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da$$

Proof

Note the following vector identity.

$$\nabla\cdot(\psi\mathbf{n}) = \psi\nabla\cdot\mathbf{n} + \mathbf{n}\cdot\nabla\psi$$

Take the dot product of **n** with the volume integral in question.

$$\begin{aligned}
\mathbf{n}\cdot\int_V \nabla\psi\, d^3x &= \int_V \mathbf{n}\cdot\nabla\psi\, d^3x \\
&= \int_V [\nabla\cdot(\psi\mathbf{n}) - \psi\nabla\cdot\mathbf{n}]\, d^3x \\
&= \int_V \nabla\cdot(\psi\mathbf{n})\, d^3x - \underbrace{\int_V \psi\nabla\cdot\mathbf{n}\, d^3x}_{=0} \\
&= \int_S (\psi\mathbf{n})\cdot\mathbf{n}\, da \\
&= \int_S \mathbf{n}\cdot(\psi\mathbf{n})\, da \\
&= \mathbf{n}\cdot\int_S \psi\mathbf{n}\, da
\end{aligned}$$

Therefore,

$$\int_V \nabla\psi\, d^3x = \int_S \psi\mathbf{n}\, da.$$