Vector Identity 20

$$\int_S \mathbf{n} \times \nabla\psi \, da = \oint_C \psi \, d\mathbf{l}$$

Proof

Start with the following vector identity.

$$\nabla \times (\psi\mathbf{n}) = \psi\nabla \times \mathbf{n} + \nabla\psi \times \mathbf{n}$$

Integrate both sides over the open surface S.

$$\int_S \nabla \times (\psi\mathbf{n}) \, da = \int_S \psi\nabla \times \mathbf{n} \, da + \int_S \nabla\psi \times \mathbf{n} \, da$$

Take the dot product of $\mathbf{n}$ with both sides. [Note: $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$]

$$\mathbf{n} \cdot \int_S \nabla \times (\psi\mathbf{n}) \, da = \mathbf{n} \cdot \int_S \psi\nabla \times \mathbf{n} \, da + \mathbf{n} \cdot \int_S \nabla\psi \times \mathbf{n} \, da$$

$$\int_S \mathbf{n} \cdot [\nabla \times (\psi\mathbf{n})] \, da = \int_S \mathbf{n} \cdot (\psi\nabla \times \mathbf{n}) \, da + \mathbf{n} \cdot \int_S \nabla\psi \times \mathbf{n} \, da$$

$$\int_S [\nabla \times (\psi\mathbf{n})] \cdot \mathbf{n} \, da = \int_S \psi\mathbf{n} \cdot (\nabla \times \mathbf{n}) \, da + \mathbf{n} \cdot \int_S \nabla\psi \times \mathbf{n} \, da$$

$$\oint_C \psi\mathbf{n} \cdot d\mathbf{l} = \int_S [\nabla \cdot (\mathbf{n} \times \psi\mathbf{n}) + \mathbf{n} \cdot (\nabla \times \psi\mathbf{n})] \, da + \mathbf{n} \cdot \int_S \nabla\psi \times \mathbf{n} \, da$$

$$0 = \int_S [\nabla \cdot \psi(\mathbf{n} \times \mathbf{n}) + (\nabla \times \psi\mathbf{n}) \cdot \mathbf{n}] \, da + \mathbf{n} \cdot \int_S \nabla\psi \times \mathbf{n} \, da$$

$$0 = \int_S (\nabla \times \psi\mathbf{n}) \cdot \mathbf{n} \, da + \mathbf{n} \cdot \int_S \nabla\psi \times \mathbf{n} \, da$$

$$0 = \oint_C \psi\mathbf{n} \cdot d\mathbf{l} + \mathbf{n} \cdot \int_S \nabla\psi \times \mathbf{n} \, da$$

$$0 = \mathbf{n} \cdot \oint_C \psi \, d\mathbf{l} + \mathbf{n} \cdot \int_S \nabla\psi \times \mathbf{n} \, da$$

Bring this second term to the left side.

$$-\mathbf{n} \cdot \int_S \nabla\psi \times \mathbf{n} \, da = \mathbf{n} \cdot \oint_C \psi \, d\mathbf{l}$$

$$\mathbf{n} \cdot \int_S \mathbf{n} \times \nabla\psi \, da = \mathbf{n} \cdot \oint_C \psi \, d\mathbf{l}$$

Therefore,

$$\int_S \mathbf{n} \times \nabla\psi \, da = \oint_C \psi \, d\mathbf{l}.$$