

Vector Identity 4

$$\nabla \times \nabla \psi = 0$$

Proof

It should be noted that the right side is the zero vector; it's not a scalar.

$$\begin{aligned}
 \nabla \times \nabla \psi &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left(\sum_{j=1}^3 \delta_j \frac{\partial}{\partial x_j} \right) \psi \\
 &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left(\sum_{j=1}^3 \delta_j \frac{\partial \psi}{\partial x_j} \right) \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 (\delta_i \times \delta_j) \frac{\partial}{\partial x_i} \frac{\partial \psi}{\partial x_j} \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \varepsilon_{ijk} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \\
 &= \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 \delta_k \varepsilon_{jik} \frac{\partial^2 \psi}{\partial x_j \partial x_i} && \text{(Let } i \text{ be } j \text{ and let } j \text{ be } i.) \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \varepsilon_{jik} \frac{\partial^2 \psi}{\partial x_j \partial x_i} && \text{(Switch the sums.)} \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \varepsilon_{jik} \frac{\partial^2 \psi}{\partial x_i \partial x_j} && \text{(Switch the differentiation order.)} \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k (-\varepsilon_{ijk}) \frac{\partial^2 \psi}{\partial x_i \partial x_j} && \text{(Switch the indices.)} \\
 &= - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \varepsilon_{ijk} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \\
 &= \mathbf{0}
 \end{aligned}$$