

Vector Identity 5

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

Proof

Let $\mathbf{A} = \mathbf{a}$.

$$\begin{aligned}
 \nabla \cdot (\nabla \times \mathbf{A}) &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \cdot \left[\left(\sum_{j=1}^3 \delta_j \frac{\partial}{\partial x_j} \right) \times \left(\sum_{k=1}^3 \delta_k A_k \right) \right] \\
 &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \cdot \left[\sum_{j=1}^3 \sum_{k=1}^3 (\delta_j \times \delta_k) \frac{\partial A_k}{\partial x_j} \right] \\
 &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \cdot \left(\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_l \varepsilon_{jkl} \frac{\partial A_k}{\partial x_j} \right) \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_i \cdot \delta_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_{il} \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{jki} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
 &= \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 \varepsilon_{jik} \frac{\partial^2 A_k}{\partial x_j \partial x_i} \quad (\text{Let } i \text{ be } j \text{ and let } j \text{ be } i.) \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{jik} \frac{\partial^2 A_k}{\partial x_j \partial x_i} \quad (\text{Switch the sums.}) \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{jik} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \quad (\text{Switch the differentiation order.}) \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 (-\varepsilon_{ijk}) \frac{\partial^2 A_k}{\partial x_i \partial x_j} \quad (\text{Switch the indices.}) \\
 &= - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
 &= 0
 \end{aligned}$$