Vector Identity 6

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

Proof

Let $\mathbf{A} = \mathbf{a}$.

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{A}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k A_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) \frac{\partial A_k}{\partial x_j}\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} \frac{\partial A_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{mil} \varepsilon_{jkl} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mj}\delta_{ik} \frac{\partial^2 A_k}{\partial x_i \partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mk}\delta_{ij} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \delta_{ik} \frac{\partial^2 A_k}{\partial x_i \partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \delta_{ij} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \frac{\partial^2 A_k}{\partial x_k \partial x_j} - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial^2 A_k}{\partial x_j \partial x_j} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \frac{\partial^2 A_k}{\partial x_j \partial x_k} - \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial^2 A_k}{\partial x_j^2} \\
&= \sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^{3} \frac{\partial A_k}{\partial x_k}\right) - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k A_k\right) \\
&= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}
\end{aligned}$$